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18.100A Analysis I  
Fall 2007

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## 18.100A Introduction to Analysis – Practice Final 3 hours

**Directions.** You can use the textbook, but no other material.

To justify the arguments, **quote theorems**, by name or number. Be sure to **verify any needed hypotheses**, unless they are standard – you don't have to prove that  $\sin x$  is continuous, for example. Work problems totalling at least 100. If you do more than that, indicate clearly which ones should not be graded, otherwise they all will be graded, but the total scaled back to 100.

1. (20: 10,10) Let  $f(x) = \sum_1^{\infty} \frac{x}{n(x+n)}$

a) Prove  $f(x)$  is continuous on  $[0, \infty)$ .

(This follows from part (b), but prove it without using part (b). Use theorems; don't go back to the definition of continuity.)

b) Prove  $f(x)$  is differentiable on  $[0, \infty)$ .

For any integer  $k \geq 0$ , express  $f'(k)$  in terms of  $k$  and  $A = f'(0)$ .

2. (15) Let  $f(x) = \sum_1^{\infty} \frac{\sin nx}{(n-1)!}$ . Prove that  $\int_0^{\pi} f(x) dx$  exists, and evaluate it. ( $0! = 1$ )

3. (15) Let  $S$  be a set of points in the plane which is closed, but not assumed to be compact.

Prove there is a point  $P$  in  $S$  which is closest to the origin  $(0, 0)$ , using the usual Euclidean distance; "closest" in the sense that no other point of  $S$  is closer to the origin.

4. (15) Let  $S$  be a compact subset of  $\mathbf{R}^2$ , and  $f(x)$  a continuous function on  $S$ . Prove that the set  $f(S)$  consisting of its values on  $S$ :

$$f(S) = \{f(x) : x \in S\}$$

is a compact subset of  $\mathbf{R}$ . (Use sequential compactness.)

5. (15)  $S(x) = \int_0^x \sin(t^2) dt$  cannot be expressed in terms of elementary functions.

Let  $f(x) = \int_0^{\pi} t^2 \cos(xt^2) dt$ . For  $a > 0$ , evaluate  $\int_0^a f(x) dx$  in terms of values of  $S(x)$ . (Refer to theorems, verify hypotheses.)

6. (20: 10, 10) a) Let  $S$  be the set whose elements are finite sequences of integers, of any length:  $S = \{\{x_1, x_2, \dots, x_n\} : x_i \in \mathbf{Z}, n \in \mathbf{N}\}$ . Prove  $S$  is a countable set.

b) Let  $S$  be the set of rational numbers in the interval  $(0, 1)$ . Prove that given  $\epsilon > 0$ , the set  $S$  is contained in a union of open intervals  $\bigcup_1^{\infty} I_n$  whose total length is  $\leq \epsilon$ .

7. (10) Let  $S_1, S_2, \dots, S_n$  be a finite set of compact intervals on the line.

Prove their union  $S = S_1 \cup S_2 \cup \dots \cup S_n$  is a sequentially compact subset of the line.

(Use the definition of sequential compactness and the methods of the earlier chapters; no credit for using or quoting any theorems from Chapter 25.)

8. (10) Assume that
- a)  $f(x)$  is continuous on  $(-\infty, \infty)$ ;
  - b)  $f(a) > 0$  for at least one point  $x = a$ ;
  - c)  $\lim_{x \rightarrow -\infty} f(x) = 0$  and  $\lim_{x \rightarrow \infty} f(x) = 0$ .

Prove that  $f(x)$  has a maximum point on  $(-\infty, \infty)$ .

9. (10) Let  $a < b$ . Prove that if a function  $f(x)$  is uniformly continuous on both of the overlapping intervals  $(-\infty, b)$  and  $(a, \infty)$ , then it is uniformly continuous on  $(-\infty, \infty)$ .  
(Go back to the definition of uniform continuity.)

10. (10) Assume that the functions  $g_n(x)$ ,  $n = 0, 1, 2, \dots$  are all continuous on  $[a, b]$ , and that on this interval they converge uniformly to  $g(x)$ .

Prove that on this interval  $[a, b]$ , their indefinite integrals also converge uniformly:

$$\int_a^x g_n(t) dt \Rightarrow \int_a^x g(t) dt.$$

(This is not difficult.)

11. (10) Let  $\phi(x) = \int_0^x f(x-t) dt$ , where  $f(x)$  is continuous for all  $x$ . Calculate  $\phi'(x)$  two ways:

- a) by Leibniz' theorem;
- b) by making a change of variable in the integral.

12. (10) Prove that  $\sum_0^{\infty} e^{-nx} \cos nx = g(x)$  sums to a function  $g(x)$  which is defined and continuous for all  $x > 0$ .