

18.100B Practice for the first midterm

Not to be turned in, just for practice.

Problems.

- 1) Let (\mathcal{M}, d) be an arbitrary metric space.
 - a) State the definition of a connected subset of \mathcal{M} .
 - b) Prove that $E \subseteq M$ is connected if and only if every non-empty proper subset has a non-empty boundary.

- 2) Let (\mathcal{M}, d) be an arbitrary metric space (e.g., not necessarily Euclidean space).
 - a) Show that a compact subset of \mathcal{M} is necessarily closed and bounded.
 - b) Give an example of a metric space with a closed and bounded subset that is *NOT* compact.

Hint: Use the discrete metric $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

- 3) Show that $\sqrt{2} + \sqrt{3}$ is irrational.
Hint: Show that $\sqrt{2} + \sqrt{3} \in \mathbb{Q} \implies \sqrt{2} \in \mathbb{Q}$.

- 4) Let (\mathcal{M}, d) be an arbitrary metric space (e.g., \mathcal{M} is not necessarily complete). If (x_n) and (y_n) are both Cauchy sequences and $d_n = d(x_n, y_n)$, show that (d_n) is a convergent sequence of real numbers.

- 5) Let (\mathcal{M}, d) be an arbitrary metric space. If $G \subseteq \mathcal{M}$ is open, and A is any subset of \mathcal{M} , show that

$$G \cap A = \emptyset \iff G \cap \overline{A} = \emptyset$$

- 6) Show that if $x, y \in \mathbb{R}$ and $x < y$ then there exists an *irrational* number between x and y . (You may use the existence of a rational number between x and y .)