18.100B Solutions to practice for the second midterm

Solutions to problems.

1) a) See Rudin's book for the definition of $\int_a^b f\, d\alpha$.

   b) If $\alpha(x) = \text{const.}$, then $\int_a^b f\, d\alpha$ exists for any bounded $f : [a, b] \to \mathbb{R}$. (The integral is, in fact, then equal to 0.)

2) Let $\varepsilon > 0$ and choose any interval of length 2, $[a, b]$. Since a continuous function on a compact set is uniformly continuous, f is uniformly continuous on $[a, b]$. Pick $0 < \delta < 1$ so that $s, t \in [a, b]$ and $|s - t| < \delta$ implies $|f(s) - f(t)| < \varepsilon$. Thanks to periodicity of f, this is actually true for any $s, t \in \mathbb{R}$ with $|s - t| < \delta$.

   Indeed, assume without loss of generality that $s < t$, we can always find $n \in \mathbb{N}$ so that both $s-n$ and $s-n+1$ are in $[a, b]$ and since $\delta < 1$, $t-n$ is also in $[a, b]$. But then $|(s - n)-(t - n)| < \delta$ implies

$$|f(s - n) - f(t - n)| = |f(s) - f(t)| < \varepsilon.$$

3) Assume that such an f fails to be monotonic. Then one of the following cases has to occur.

   a) $f(x) > f(y)$ and $f(y) < f(z)$, for some $x < y < z$.

   b) $f(x) < f(y)$ and $f(y) > f(z)$, for some $x < y < z$.

   By the continuity of f, it maps connected sets to connected sets so both $f((x, z))$ and $f([x, z])$ are intervals and since it also maps compact sets to compact sets, and $f((x, z)) \subseteq f([x, z])$ these intervals are bounded. So, let's say

$$f((x, z)) = (a, b)$$

   for some real numbers $a < b$.

   Suppose now that case a) above holds. Then, by continuity, f has to attain a minimum on $[x, z]$, say m and, by a), this minimum is attained in the interior (x, z), so $m \in f((x, z))$ but it is not an interior point of $f((x, z))$ (as there are no values below it)! Also, if case b) holds, we conclude that f attains its maximum on $[x, z]$ in its interior (x, z), which contradicts that $f((x, z))$ has to be an open interval.

4) Suppose that f attains both values 1 and 2. Then $A = f^{-1}(\{1\})$ and $B = f^{-1}(\{2\})$ are non-empty. By continuity of $f : X \to Y$, the sets A and B are open in X, since $\{1\}$ and $\{2\}$ are open in Y. Hence we conclude $X = A \cup B$, where A and B are open, disjoint, and non-empty sets. This contradicts the connectedness of X.

   (With respect to Rudin's terminology, we remark that two disjoint, open sets A and B are always separate, i. e., we have $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, since limit points of A cannot be interior points B and vice versa.)

5) Since f and α are increasing functions that are discontinuous at the the same $x_0 \in [a, b]$, we conclude that

$$f(y) - f(x) \geq A \quad \text{and} \quad \alpha(y) - \alpha(x) \geq B$$

whenever $x < x_0 < y$, where

$$A = f(x_0+) - f(x_0-) > 0 \quad \text{and} \quad B = \alpha(x_0+) - \alpha(x_0-) > 0.$$

Any partition P of $[a, b]$ contains to consecutive points $x_{i-1} < x_i$ such that $x_{i-1} < x_0 < x_i$, which leads to

$$U(P, f, \alpha) - L(P, f, \alpha) \geq [f(x_i) - f(x_{i-1})][\alpha(x_i) - \alpha(x_{i-1})] > AB > 0.$$

Thus we cannot have $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ when $0 < \epsilon \leq AB$, which shows that $f \notin \mathcal{R}(\alpha)$.

6) Consider $h(x) = g(x)f(x)$ with some $g(x)$ continuous on $[a, b]$ and differentiable on (a, b). Then $h(a) = h(b) = 0$ and, by Rolle's theorem (or, more generally, by the mean-value theorem)

$$h'(c) = g'(c)f(c) + g(c)f'(c) = 0,$$

for some $c \in (a, b)$. Taking $g(x) = e^{-\lambda x}$ we get $f'(c) = \lambda f(c)$ (regardless of who f is). One way of coming up with this g is to note that since for $f'(c) = \lambda f(c)$ to hold, it suffices to have that $g'(x)/g(x) = -\lambda$, or $g'(x) = -\lambda g(x)$, which is solved by $g(x) = e^{-\lambda x}$.