SOLUTIONS TO PS 8

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Solution/Proof of Problem 1. *From* $f(x) = f(x^2)$, *we have*

$$f(x) = f(x^{\frac{1}{2}}) = f(x^{\frac{1}{4}}) = \cdots = f(x^{\frac{1}{2n}}).$$

Now let $y_n = x^{\frac{1}{2n}}$, *and assume* $x \neq 0$, *so* $\lim_{n\to\infty} y_n = 1$. *Since* f *is continuous, we have if* $x \neq 0$

$$f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} f(y_n) = f(\lim_{n\to\infty} y_n) = f(1).$$

When $x = 0$, *then* $f(0) = \lim_{y\to 0} f(y) = f(1)$.

Then f *is a constant.*

Solution/Proof of Problem 2. *From MVT, we have* $\forall x > 0, \exists y = y(x) \in (x, x+1)$, *s.t.*

$$g(x) = f(x+1) - f(x) = f'(y).$$

Notice that $y > x$, *so* $\lim_{x\to\infty} y = \infty$, *so we have*

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f'(y) = 0,$$

since $\lim_{y\to\infty} f'(y) = 0$.

Solution/Proof of Problem 3. *Consider the function*

$$g(x) = C_0 x + \frac{C_1 x^2}{2} + \cdots + \frac{C_n x^{n+1}}{n+1}.$$

Then $g(0) = 0$ *and* $g(1) = C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n+1} = 0$. *By the mean value theorem, we have that* $g'(y) = 0$ *for some* $y \in (0,1)$, *which means the equation* $C_0 + C_1 x + \cdots + C_n x^n$ *has a root between* 0 *and* 1.

Solution/Proof of Problem 4. *(a) Suppose* f *has two fixed point,* $x_1 < x_2$. *Then by MVT we have that*

$$\exists y \in (x_1, x_2), \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(y).$$

Then $f'(y) = 1$ *which is a contradiction.*

(b) If f *has a fixed point, then*

$$f(t) = t \;\Rightarrow\; t = t + (1+e^t)^{-1} \;\Rightarrow\; (1+e^t)^{-1} = 0.$$

But we know that $e^t > 0$ *then* $(1+e^t)^{-1} \neq 0$. *So* f *has no fixed point.*

(c)Consider a sequence defined by $x_n = f(x_{n-1})$ *for any* $x_1 \in \mathbb{R}$. *Then we have*

$$|x_n - x_{n-1}| = |f(x_{n-1}) - f(x_{n-2})| = |f'(y)||x_{n-1} - x_{n-2}| \leqslant A|x_{n-1} - x_{n-2}|.$$

By using $|x - z| \leqslant |x - y| + |y - z|$, *we have*

$$|x_n - x_m| \leqslant |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \leqslant \left(\sum_{i=m-1}^{n-2} A^i\right) |x_2 - x_1|.$$

Since $A < 1$, the series $\sum A^i$ converges, and so the partial sums form a Cauchy sequence. This inequality shows that $\{x_n\}$ is also a Cauchy sequence and hence converges.

Notice that $|f(x+h) - f(x)| \leqslant Ah$, so f is continuous. Then we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(\lim_{n\to\infty} x_{n-1}) = f(x).$$

So x is a fixed point of $f(x)$.

Solution/Proof of Problem 5. *$f'(0)$ exists because of the following:*

By definition, $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x}$. Since we have $\frac{(f(x)-f(0))'}{(x)'} = f'(x)$, and the limit $\lim\limits_{x\to 0} f'(x) = 3$ exists, from L'Hospital rule, we know $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x}$ exists and $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x} = 3$.

Solution/Proof of Problem 6. *From Taylor's theorem, we have*

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 \\ &\quad + \cdots + \frac{1}{(n-1)!} f^{(n-1)}(x_0)(x - x_0)^{n-1} + \frac{1}{n!} f^{(n)}(y)(x - x_0)^n, \end{aligned}$$

for some $y \in (x, x_0)$ or (x_0, x). Then we have

$$f(x) = f(x_0) + \frac{1}{n!} f^{(n)}(y)(x - x_0)^n \quad \Rightarrow \quad f^{(n)}(y) = n! \frac{f(x) - f(x_0)}{(x - x_0)^n}.$$

We want to say that $f^{(n)}(y)$ has the same sign as $f^{(n)}(x_0)$, but we have to be careful because $f^{(n)}$ need not be continuous. It may well be the case that there is a sequence z_n approaching x_0 such that $f^{(n)}(z_n)$ does not go to $f^{(n)}(x_0)$. Nevertheless, we will show that if $y(x)$ is the intermediate point in (x, x_0) appearing in Taylors's theorem, then as $x \to x_0$, $f^{(n)}(y) \to f^{(n)}(x_0)$.

The reason is that, as pointed out above,

$$\lim_{x\to x_0} f^{(n)}(y) = \lim_{x\to x_0} n! \frac{f(x) - f(x_0)}{(x - x_0)^n},$$

and we can compute the limit on the right by using L'Hôpital's rule $n - 1$ times:

$$\lim_{x\to x_0} n! \frac{f(x) - f(x_0)}{(x - x_0)^n} = \lim_{x\to x_0} (n-1)! \frac{f'(x)}{(x - x_0)^{n-1}} = \ldots = \lim_{x\to x_0} \frac{f^{(n-1)}(x)}{(x - x_0)}$$

and then noticing that since $f^{(n-1)}(x_0) = 0$, this is equal to

$$\lim_{x\to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{(x - x_0)} = f^{(n-1)}(x_0) = A$$

If $A > 0$, then $\frac{f(x)-f(x_0)}{(x-x_0)^n} > 0$ in a neighborhood of x_0. If n even, this implies $f(x) - f(x_0) > 0$ for any element in the neighborhood of x_0, i.e x_0 is a local minimum. When n is odd, f does not have a local minimum or maximum.

Similarly, if $A < 0$ and n even, then $f(x) - f(x_0) < 0$ for any element in the neighborhood of x_0, i.e x_0 is a local maximum. When n is odd, f does not have a local minimum or maximum.

Solution/Proof of Problem 7. *For $x > 0$ we have $f(x) = x^3$ and hence $f'(x) = 3x^2$ and $f''(x) = 6x$. For $x < 0$ we have $f(x) = -x^3$ and hence $f'(x) = -3x^2$ and $f''(x) = -6x$. Notice that we can write, for $x \neq 0$,*

$$f(x) = |x|x^2, \quad f'(x) = 3|x|x, \quad f''(x) = 6|x|.$$

Hence at $x = 0$ we have

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|h^2}{h} = \lim_{h\to 0} |h|h = 0 \\ f''(0) &= \lim_{h\to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h\to 0} \frac{3|h|h}{h} = \lim_{h\to 0} 3|h| = 0 \\ f'''(0) &= \lim_{h\to 0} \frac{f''(h) - f''(0)}{h} = \lim_{h\to 0} \frac{6|h|}{h} \text{ does not exist} \end{aligned}$$