

18.100B, FALL 2002
IN-CLASS TEST 1: OCTOBER 17

Try each of the questions; they will be given equal value even though some are longer (or harder) and some less so. You may use theorems from class, or the book, provided you can recall them correctly! No books or papers are permitted.

You will likely get away with less explanation than I am giving here. I will add some comments on your solutions when I have seen them.

PROBLEM 1

Let $\{x_n\}$ be a sequence of positive real numbers such that $x_n^3 > n$. Show that $1/x_n \rightarrow 0$ as $n \rightarrow \infty$ with respect to the usual metric on \mathbb{R} .

Solution. The sequence $y_n = 1/x_n$ satisfies $0 < y_n^3 < 1/n$. Given $\epsilon > 0$ choose $N > 1/\epsilon^3$, then $n > N$ implies $y_n^3 < 1/n < 1/N < \epsilon^3$ so $y_n < \epsilon$ and it follows that $y_n \rightarrow 0$ as $n \rightarrow \infty$. \square

PROBLEM 2

Consider the metric space which is the subset $E = \{0\} \cup \bigcup_{n \in \mathbb{N}} \{1/n\}$ of the real numbers with the metric induced by the usual metric on \mathbb{R} .

- (1) What is the set E' of limit points of E , in \mathbb{R} ?
- (2) Describe all the closed subsets of E .
- (3) Describe all the compact subsets of E .
- (4) Describe all the connected subsets of E .

In each case justify your answer.

Solution. (1) 0 is the only limit point. It is a limit point since $1/n \rightarrow 0$ as $n \rightarrow \infty$ and all other points are isolated, so cannot be limit points.

- (2) A subset $C \subset E$ is closed if it is finite or contains 0 (or both). To see this, recall that C is closed in E if and only if it contains all its limit points. A limit point of C must be a limit point of E so the only possibilities are that C has no limit points or it contains 0. In the latter case it is closed and the former means that 0 is either not in C and is not a limit point of C , which therefore must be finite.
- (3) Since E contains its only limit point in \mathbb{R} it is closed and bounded in \mathbb{R} , so it is compact by the Heine-Borel Theorem. Thus the compact subsets of E are just the closed sets as described above, since as shown in class a subset of a compact space is compact if and only if it is closed.
- (4) Let $G \subset E$ be a connected set. If $p = 1/n \in G$ is one of the isolated points of E then $G = A \cup B$, $A = \{p\}$, $B = G \setminus \{p\}$ is a decomposition with $\bar{A} = A$ and $p \notin \bar{B}$ so $\bar{A} \cap B = \emptyset$ and $\bar{A} \cap B = \emptyset$. Thus $B = \emptyset$ by the definition of connectedness and $G = \{p\}$. Thus the only possibility is that G consists of any one point of E and these sets are trivially connected since in any decomposition $\{q\} = A \cup B$ with $A \cup B = \emptyset$ one of A or B must be empty. \square

PROBLEM 3

Prove that in any metric space, a finite union of compact subsets is compact.

Solution. Let K_i , $i = 1, \dots, N$ be a finite collection of compact sets in a metric space X and let $K = \bigcup_i K_i$ be their union. An open cover U_a , $a \in A$ of K covers each of the K_i . By the assumed compactness there is a finite subcover $U_{a_{i,j}}$ such that $K_i \subset \bigcup_{j=1}^{N(i)} U_{a_{i,j}}$ for each i . All these sets taken together give a finite subcover of K which is therefore compact. \square

PROBLEM 4

Suppose that x and y are two points in the unit disk $D = \{|z| < 1; z \in \mathbb{R}^2\}$ in \mathbb{R}^2 . Using results from class (or otherwise) show that the set $\{tx + (1-t)y; t \in [0, 1]\}$ is connected. Using this, or otherwise, show that if D has a decomposition $D = A \cup B$ where $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$ (closure in D) then one of them must be empty. What does this say about D ?

Solution. For two fixed points $x, y \in D$, the map $f : [0, 1] \rightarrow tx + (1-t)y$ is continuous since each component is a linear function of t , hence continuous. By theorems from class, the interval $[0, 1]$ is connected and the image of a connected set under a continuous map is connected. Thus the line $L_{x,y} = \{tx + (1-t)y; t \in [0, 1]\}$ is connected. It lies in D by the triangle inequality

$$|tx + (1-t)y| \leq t|x| + (1-t)|y| < 1.$$

Now, suppose that $D = A \cup B$ is a decomposition as described above and that neither A nor B is empty. Thus we can choose $x \in A$ and $y \in B$. Now consider the decomposition of $L_{x,y} = A' \cup B'$ where $A' = A \cap L_{x,y}$ and $B' = B \cap L_{x,y}$. Since $L_{x,y}$ is the continuous image of a compact set it is also compact, and hence closed. Thus the closures in $L_{x,y}$ satisfy $\bar{A}' \subset \bar{A} \cap L_{x,y}$ and $\bar{B}' \subset \bar{B} \cap L_{x,y}$. Hence we deduce that $\bar{A}' \cap B' = A' \cap \bar{B}' = \emptyset$. So using the connectedness of $L_{x,y}$ we deduced that one of A' or B' must be empty, but this contradicts the assumption that both A and B are non-empty. Thus one of A or B must in fact be empty and this means that D itself must be connected. \square

Remark: What you are showing here is that ‘pathwise connectedness implies connectedness’.