SOLUTIONS TO PS6

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Solution/Proof of Problem 1. *Since* $\sum |a_n|$ *converges,* $\lim\limits_{n\to\infty} |a_n| = 0$*. So* $\exists N \in \mathbb{N}$ *such that for* $n \geqslant N$*,* $|a_n| < 1$*. Thus for* $n \geq N$ *we have* $|a_n^2| \leq |a_n|$ *and by the comparison theorem and the convergence of* $\sum |a_n|$ *we conclude that* $\sum |a_n^2|$ *converges.*

Solution/Proof of Problem 2. *Notice that*

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right),$$

so

$$\begin{aligned} \sum_{n=1}^{m} \frac{1}{n(n+1)(n+2)} &= \sum_{n=1}^{m} \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(m+1)(m+2)}\right). \end{aligned}$$

Then $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.

Solution/Proof of Problem 3.

a) $\sum\limits_{k=1}^{n} a_k = \sum\limits_{k=1}^{n} (\sqrt{k+1} - \sqrt{k}) = \sum\limits_{k=1}^{n} \sqrt{k+1} - \sum\limits_{k=1}^{n} \sqrt{k} = \sum\limits_{k=2}^{n+1} \sqrt{k} - \sum\limits_{k=1}^{n} \sqrt{k} = \sqrt{n+1} - 1$*, so diverges.*

b) Notice that

$$\frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \frac{1}{2n\sqrt{n}}$$

and $\sum \frac{1}{2n\sqrt{n}}$ *converges, hence by the comparison theorem so does* $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n}$

c) When $\alpha > 1$*, we have* $\sum\limits_{k=1}^{n} \frac{1}{\alpha^k}$ *converges. So* $\sum\limits_{k=1}^{n} \frac{1}{1+\alpha^n}$ *is an increasing bounded sequence, so it converges.*

When $\alpha \leqslant 1$*, we have* $\sum\limits_{k=1}^{n} \frac{1}{1+\alpha^k} \geqslant \sum\limits_{k=1}^{n} \frac{1}{2} = \frac{n(n+1)}{2}$*, so* $\sum\limits_{k=1}^{n} \frac{1}{1+\alpha^n}$ *diverges.*

Solution/Proof of Problem 4. *The Cauchy-Schwarz inequality tells us that*

$$\left|\sum_{k=1}^{n} \sqrt{a_k}\frac{1}{k}\right| \leq \left(\sum_{k=1}^{n} (\sqrt{a_k})^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \left(\frac{1}{k}\right)^2\right)^{\frac{1}{2}}$$

and since both $\sum a_k$ *and* $\sum \frac{1}{k^2}$ *converge, we know that* $\sum_{k=1}^{n} \frac{\sqrt{a_k}}{k}$ *is a bounded increasing sequence, hence it must converge.*

Solution/Proof of Problem 5. *Notice that* $\sum a_n$ *converges so* $\lim\limits_{n\to\infty} a_n = 0$ *and since* a_n *is decreasing this implies* $a_n \geq 0$.

Now since

$$\sum_{k=n+1}^{2n} a_k \geqslant n a_{2n} \geqslant 0,$$

take limits on both side, we have $\lim_{n\to\infty} n a_{2n} = 0$ *i.e.* $\lim_{2n\to\infty} 2n a_{2n} = 0$. *Similarly, we can prove* $\lim_{2n+1\to\infty} (2n+1) a_{2n+1} = 0$. *So*

$$\lim_{n\to\infty} n a_n = 0.$$

Solution/Proof of Problem 6.

1) *First note that, for any function* f *and any set* $B \subseteq Y$, *it is always true that* $X = f^{-1}(B) \cup f^{-1}(B^c)$, *and since these sets are disjoint, we always have* $f^{-1}(B^c) = f^{-1}(B)^c$.

   (a) $\iff$ **(b)**

   Suppose (a) is true and B *is closed in* Y, *then* B^c *is open in* Y *and (a) implies* $f^{-1}(B^c) = f^{-1}(B)^c$ *open in* X *and hence* $f^{-1}(B)$ *is open in* X. *This proves (a)* $\implies$ *(b), and exchanging the words open and closed we get a proof that (b)* $\implies$ *(a).*

2) *Note that for any function* f *and any sets* $A \subseteq X$, $B \subseteq Y$ *we always have*

$$A \subseteq f^{-1}(f(A)), \qquad f\left(f^{-1}(B)\right) \subseteq B.$$

   The first inclusion is an equality precisely when f *is one-to-one, while the second is an equality precisely when* f *is onto. (So for instance if* f *is constant and* X *and* Y *have more than one point, both inclusions are strict in general.)*

   (b) $\implies$ **(c)**

   Suppose (b) is true and let A *be any subset of* X. *We know that*

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\left(\overline{f(A)}\right)$$

   and that $f^{-1}\left(\overline{f(A)}\right)$ *is closed, hence* $\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$ *and so*

$$f\left(\overline{A}\right) \subseteq f\left(f^{-1}\left(\overline{f(A)}\right)\right) \subseteq \overline{f(A)}$$

   and (c) is true.

   (c) $\implies$ **(b)**

   Now suppose (c) is true, and $B \subseteq Y$ *is closed. We can apply (c) to* $A := f^{-1}(B)$ *and get*

$$f\left(\overline{A}\right) \subseteq \overline{f(A)} \iff f\left(\overline{f^{-1}(B)}\right) \subseteq \overline{f\left(f^{-1}(B)\right)}$$

   As mentioned above, $f\left(f^{-1}(B)\right) \subseteq B$ *so we have*

$$f\left(\overline{f^{-1}(B)}\right) \subseteq \overline{f\left(f^{-1}(B)\right)} \subseteq \overline{B} = B$$

   hence

$$\overline{f^{-1}(B)} \subseteq f^{-1}\left(f\left(\overline{f^{-1}(B)}\right)\right) \subseteq f^{-1}(B)$$

   which implies $f^{-1}(B)$ *is closed, i.e. (b) is true.*

Solution/Proof of Problem 7. *Consider the function*

$$f(x) = \begin{cases} 0 & \textit{if } x = 0 \\ 1 & \textit{if } x \neq 0 \end{cases}$$

Then it is easy to see that $f(x)$ *is not continuous at the point* $x = 0$*. But it satisfies the condition in the problem. So that condition does* **NOT** *imply* f *continuous.*