18.100B Practice for the second midterm

Not to be turned in, just for practice.

Problems.

1) Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and let $\alpha : [a, b] \to \mathbb{R}$ be monotonically increasing.
   a) State the definition of the Riemann–Stieltjes integral
   $$\int_a^b f \, d\alpha$$
   b) Give an example for α such that $\int_a^b f \, d\alpha$ exists for any bounded $f : [a, b] \to \mathbb{R}$.

2) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic of period 1, i. e., we have $f(x + 1) = f(x)$ for every $x \in \mathbb{R}$. Prove that f is uniformly continuous.

3) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that $f(V) \subseteq \mathbb{R}$ is open for every open set $V \subseteq \mathbb{R}$. Prove that f has to be a monotonic function.

4) Let X be any metric space, and consider the metric space $Y = \{1, 2\}$ equipped with the discrete metric (i. e., $d_Y(p, q) = 1$ if $p \neq q$ and $d_Y(p, p) = 0$). Show that if $f : X \to Y$ is continuous and X is connected, then f has to be constant.

5) Let $f : [a, b] \to \mathbb{R}$ and $\alpha : [a, b] \to \mathbb{R}$ be monotonically increasing functions. Prove that if f and α are discontinuous at the same $x_0 \in [a, b]$, then $\int_a^b f \, d\alpha$ does not exist.

6) Assume $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) = f(b) = 0$, and suppose $f'(x)$ exists on (a, b). Prove that for every real λ there is some $c \in (a, b)$ such that $f'(c) = \lambda f(c)$.
   Hint: Apply the mean-value theorem to $g(x)f(x)$ for a suitable g depending on λ.