18.100B Problem Set 2

Due Friday September 22, 2006 by 3 PM

Problems:

1) (10 pts) Prove that the empty set is a subset of every set.

2) (10 pts) If x, y are complex, prove that
$$||x| - |y|| \leq |x - y|.$$
(*Hint:* This is equivalent to proving the following two inequalities: $|x| \leq |x - y| + |y|$ and $|y| \leq |x - y| + |x|$. Why?)

3) (10 pts) Find $\sup M$ and $\inf M$ for:
   a) $M = \left\{ \frac{|x|}{1+|x|} : x \in \mathbb{R} \right\}$,
   b) $M = \left\{ \frac{x}{1+x} : x > -1 \right\}$,
   c) $M = \left\{ x + \frac{1}{x} : \frac{1}{2} < x < 2 \right\}$.

4) (10 pts) Let:
   a) S be the set of all natural numbers that are not divisible by a square number;
   b) T be the set of all natural numbers that have exactly three prime divisors;
   c) U be the set of all natural numbers that are less or equal than 200.
   Determine $S \cap T \cap U$ explicitly.

5) (10 pts) Let X and Y be two disjoint sets. Suppose further that $X \sim \mathbb{R}$ and that $Y \sim \mathbb{N}$ (i. e. the set Y is countable). Show that $Z = X \cup Y$ satisfies $Z \sim \mathbb{R}$.

6) (10 pts) Construct a bounded set of real numbers with exactly three limit points. In addition, construct a bounded set of real numbers with countably many limit points.

7) (10 pts) Let E be a subset of a metric space. The *interior* E° is defined by
$$E^\circ = \{x \in E : x \text{ is an interior point}\}.$$
   a) Prove that E° is always open.
   b) Prove that E is open if and only if $E^\circ = E$.
   c) If $G \subseteq E$ and G is open, prove that $G \subseteq E^\circ$.

Extra problems:

1) Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ with $f(z) = 1/z$. Sketch the following sets in the complex plane:
   a) $f(\mathbb{R} \setminus \{0\})$,
   b) $f(B_r)$ where $B_r = \{z \in \mathbb{C} : |z| = r\}$ and $r > 0$,
   c) $f(i\mathbb{R} \setminus \{0\})$,
   d) $f(A)$ where $A = \{z \in \mathbb{C} : \operatorname{Re} z = 1\}$.

   [Recall that, for a given function $f : X \to Y$, the set $f(E) = \{f(x) : x \in E\}$ is the *image* of a subset $E \subseteq X$ under f.]

2) A complex number z is said to be *algebraic* if there are integers $a_0, \dots, a_n$, not all zero, such that
$$a_0 z^n + a_1 z^n + \cdots + a_{n-1} z + a_n = 0.$$
   Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with
$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

3) If you think of the existence of a 1-1 map from A into B as saying that A is 'not bigger than' B (think $\leq$). Then this exercise proves that: if A is not bigger than B and B is not bigger than A, then A and B are the same size.
   Prove the Schroeder-Bernstein theorem

   > If A and B are any two sets, f is a 1-1 map from A *into* B and g is a 1-1 map from B *into* A, then there exists a map F from A to B which is 1-1 and onto, i.e., $A \sim B$.

   by the following steps (due to Birkhoff and MacLane):
   i) Define 'ancestors' as follows: Let $a \in A$, if $a \in g(B)$ then we call $g^{-1}(a)$ the first ancestor of a (we call a itself the zero$^{\text{th}}$ ancestor of a). If $g^{-1}(a)$ is in $f(A)$ then we call $f^{-1}\left(g^{-1}(a)\right)$ the second ancestor of a. If this is in the image of g, then we call $g^{-1}\left(f^{-1}\left(g^{-1}(a)\right)\right)$ the third ancestor of a and so on.
   Show that this divides A into three disjoint subsets: A_∞ made up of the elements that have infinitely many ancestors, A_e made up of the elements that have an even number of ancestors, and A_o made up of the elements that have an odd number of ancestors.
   ii) Show that you can partition B into three similar subsets: B_∞, B_e, and B_o.
   iii) Identify $f(A_\infty)$, $f(A_e)$, and $f(A_o)$.
   iv) Define
$$F(a) = \begin{cases} f(a) & \text{if } a \in A_\infty \cap A_e \\ g^{-1}(a) & \text{if } a \in A_o \end{cases}$$
   and show that F is a 1-1 correspondence between A and B.

4) Show that if $Sq = [0,1] \times [0,1]$ is the unit square and $I = [0,1]$ is one of its sides, then $Sq \sim I$.