18.100B, FALL 2002
PRACTICE TEST 1

Try each of the questions; they will be given equal value. You may use theorems from class, or the book, provided you can recall them correctly!

Problem 1

Consider the set S defined as follows. The elements of S are sequences, $\{s_n\}_{n=1}^{\infty}$ with all entries either 1 or 2 and with the additional property that every 2 is followed by a 1. Said more precisely, for every n, $s_n = 1$ or $s_n = 2$ and if $s_n = 2$ then $s_{n+1} = 1$. Say why precisely one of the following is true
(a) S is finite
(b) S is countably infinite
(c) S is uncountably infinite
and then decide which one is true and *prove* it.

Problem 2

Consider the metric space $M = [0, 1] = \{x \in \mathbb{R}; 0 \le x \le 1\}$ with the usual metric, $d(x, y) = |x - y|$. Is the set $A = [0, \frac{1}{2}) = \{x \in \mathbb{R}; 0 \le x < \frac{1}{2}\}$ open as a subset of M? What is the closure of A as a subset of M? Is A compact? Is the closure of A compact? In each case justify your answer.

Problem 3

Let M be a *compact* metric space. Suppose $A \subset M$ is *not* compact. Show, directly from the definition or using a theorem proved in class, that A is *not* closed.

Problem 4

Recall that a set S in a metric space M is connected if any separated decomposition of it, $S = A \cup B$ where $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, is 'trivial' in the sense that either A or B is empty. Show that the whole metric space M is connected if and only if the only subsets $A \subset M$ of it which are *both open and closed* are the 'trivial' cases $A = \emptyset$ and $A = M$.

Another possible Test

Problem 1

Show that the set $\{0\} \cup \{1/n; n \in \mathbb{N}\}$ is compact as a subset of the metric space $\mathbb{Q}$, the rational numbers, with the usual metric $d(x, y) = |x - y|$.

Problem 2

Let X be a set with the discrete metric, $d(x, x) = 0$ and $d(x, y) = 1$ if $x \ne y$. Show that every function $f : X \longrightarrow \mathbb{C}$ is continuous.

Problem 3

Consider the metric $d(x, y) = d_1((x_1, x_2)(y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ on $\mathbb{R}^2$.

(1) Show that if d is the usual metric on $\mathbb{R}^2$ then

$$d(x, y) \leq d_1(x, y) \leq 2d(x, y) \ \forall \ x, y \in \mathbb{R}^2.$$

(2) Show that the open sets relative to d_1 are the same as those relative to d.

Problem 4

Suppose that X is a metric space and $A \subset X$ is an open set which is compact and is neither empty nor equal to X. Show that X is not connected.