

# 18.100B Problem Set 3 Solutions

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- 1) We begin by defining  $d : V \times V \rightarrow \mathbb{R}$  such that  $d(x, y) = \|x - y\|$ . Now to show that this function satisfies the definition of a metric.  $d(x, y) = \|x - y\| \geq 0$  and

$$d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$$

So the function is positive definite.

$$d(x, y) = \|x - y\| = \| -1(y - x) \| = | -1 | \|y - x\| = \|y - x\| = d(y, x)$$

Thus the function is symmetric. Finally,

$$d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$$

So the triangle inequality holds. Therefore  $d$  is a metric.

- 2) Once again we must verify the properties of a metric. We have defined  $d_1$  as

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Since  $d$  is a metric, it only takes nonnegative values, so  $d_1$  cannot be negative.  $d_1(x, y)$  is zero exactly when  $d(x, y)$  is, so only for  $x = y$ . Therefore  $d_1$  is positive definite. Since  $d$  is symmetric,  $d_1$  obviously inherits this property. Finally, for  $x, y, z \in M$

$$\begin{aligned} d_1(x, y) + d_1(y, z) &= \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} = \frac{d(x, y) + d(y, z) + 2d(x, y)d(y, z)}{1 + d(x, y) + d(y, z) + d(x, y)d(y, z)} \\ &\geq \frac{d(x, y) + d(y, z) + d(x, y)d(y, z)}{1 + d(x, y) + d(y, z) + d(x, y)d(y, z)} = 1 - \frac{1}{1 + d(x, y) + d(y, z) + d(x, y)d(y, z)} \\ &\geq 1 - \frac{1}{1 + d(x, y) + d(y, z)} \geq 1 - \frac{1}{1 + d(x, z)} = \frac{d(x, z)}{1 + d(x, z)} = d_1(x, z) \end{aligned}$$

So the triangle inequality holds, thus we have a metric. It is easy to see that this metric never takes on a value larger than 1, since  $d(x, y) < 1 + d(x, y)$ , so under the metric  $d_1$ ,  $M$  is bounded.

- 3) a)  $A, B \subseteq M$ ,  $M$  a metric space. Suppose  $x \in A^\circ \cup B^\circ$ . Without loss of generality, say  $x \in A^\circ$ . Therefore  $x$  is an interior point of  $A$ , so  $\exists \epsilon > 0$  such that the ball of radius  $\epsilon$  centered at  $x$  is contained in  $A$ , or  $B_\epsilon(x) \subseteq A$ . Since  $A \subseteq A \cup B$ ,

$$B_\epsilon(x) \subseteq A \cup B \implies x \in (A \cup B)^\circ$$

This shows that  $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ .

- b) Now let  $x \in A^\circ \cap B^\circ$ . Therefore  $x \in A^\circ$ , so  $x$  is an interior point of  $A$ , hence  $\exists \epsilon_1 > 0$  such that  $B_{\epsilon_1}(x) \subseteq A$ . Similarly,  $x \in B^\circ \implies \exists \epsilon_2 > 0$  such that  $B_{\epsilon_2}(x) \subseteq B$ . Let  $\delta = \min(\epsilon_1, \epsilon_2)$ . By the triangle inequality,

$$\delta \leq \epsilon_i \implies B_\delta(x) \subseteq B_{\epsilon_i}(x) \implies B_\delta(x) \subseteq A \text{ and } B_\delta(x) \subseteq B.$$

Therefore  $B_\delta(x) \subseteq A \cap B$ , so  $x$  is an interior point of  $A \cap B$ . Hence  $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$ .

Let  $x \in (A \cap B)^\circ$ . So  $\exists \varepsilon > 0$  with  $B_\varepsilon(x) \subseteq A \cap B$ . Therefore  $B_\varepsilon(x) \subseteq A$  so  $x \in A^\circ$ , and similarly  $x \in B^\circ$ . So  $x \in A^\circ \cap B^\circ$ . Thus  $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$ . So these two sets are equal.

Let  $A = (-1, 0]$  and  $B = [0, 1)$ . Then 0 is an interior point of neither  $A$  nor  $B$ , so  $0 \notin A^\circ \cup B^\circ$ . But  $A \cup B = (-1, 1)$ , so  $0 \in (A \cup B)^\circ$ . Therefore in this instance the two sets are unequal.

- 4) a) If  $x \in \partial A$  then every ball around  $x$  intersects  $A$  and  $A^c$ . Thus  $x \in \overline{A}$  and  $x$  is a limit point of  $A^c$  or  $x \in A^c$  and  $x$  is a limit point of  $A$ . Either way,  $x \in \overline{A} \cap \overline{A^c}$ , and hence  $\partial A \subseteq \overline{A} \cap \overline{A^c}$ .

Now let  $x \in \overline{A} \cap \overline{A^c}$ . Since  $x \in \overline{A}$ , either  $x \in A$  or  $x$  is a limit point of  $A$ , and in both cases any open ball around  $x$  intersects  $A$ . Similarly,  $x \in \overline{A^c}$  implies any open ball around  $x$  intersects  $A^c$ . Therefore  $x \in \partial A$ , so  $\overline{A} \cap \overline{A^c} \subseteq \partial A$ . So these two sets are equal.

- b) Let  $p \in \partial A$ . By a),  $p \in \overline{A}$ . Suppose  $p \in A^\circ$  then  $\exists \varepsilon > 0$  such that  $B_\varepsilon(p) \subseteq A$ . But this is an open ball centered at  $p$  which does not intersect  $A^c$ , so  $p \notin \partial A$ . This contradiction implies that  $p \notin A^\circ$ .

Now suppose  $p \in \overline{A} \setminus A^\circ$ . For any  $\varepsilon > 0$ ,  $p \in \overline{A}$  gives that  $B_\varepsilon(x)$  intersects  $A$ , and  $p \notin A^\circ$  implies that  $B_\varepsilon(x) \not\subseteq A$ , so  $B_\varepsilon(x)$  intersects  $A^c$ . So  $p \in \partial A$ , and this shows that  $\partial A = \overline{A} \setminus A^\circ$ .

- c) By a),  $\partial A$  can be written as the intersection of two closed sets. Thus  $\partial A$  is closed.

- d) Suppose  $A$  is closed. Then  $\overline{A} = A$ , so by a)

$$\partial A = \overline{A} \cap \overline{A^c} = A \cap \overline{A^c} \subseteq A$$

Conversely, note that for any set  $B$ , if  $x \notin B$  and  $x \notin \partial B$ , then there is a positive  $r > 0$  such that  $B_r(x) \subseteq B^c$  and hence  $x \notin \overline{B}$ . This implies that

$$\text{for any set } B, \overline{B} \subseteq B \cup \partial B.$$

So if  $\partial A \subseteq A$ , then  $\overline{A} \subseteq A \cup \partial A = A \subseteq \overline{A}$  i.e.,  $A = \overline{A}$  hence  $A$  is closed.

- 5) We will show that  $S_r(x) := \{y : d(x, y) = r\}$  is the boundary of  $B_r(x)$ . It will follow from the previous exercise that

$$\overline{B_r(x)} = \partial B_r(x) \cup B_r(x) = \{y : d(x, y) \leq r\}.$$

It is clear that if  $y$  is such that  $d(x, y) = r$  then  $y \in \partial B_r(x)$  since any ball around  $y$  will have points that are closer to  $x$  and points that are further away. We just have to show that if  $d(x, y) \neq r$ , then  $y$  is not in  $\partial B_r(x)$ .

But if  $d(x, y) < r$  then for any  $0 < \varepsilon < r - d(x, y)$  the ball of radius  $\varepsilon$  around  $y$  is all inside  $B_r(x)$  and  $y \notin \partial B_r(x)$ ; and if  $d(x, y) > r$  then for any  $0 < \delta < d(x, y) - r$  the ball of radius  $\delta$  around  $y$  is all outside of  $B_r(x)$  so that again  $y \notin \partial B_r(x)$ . Thus  $\partial B_r(x)$  is precisely  $S_r(x)$  and we are done.

Here is an example of a different metric space where this result is not true: Consider  $\mathbb{R}^n$  with the discrete metric,

$$\tilde{d}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

and the ball around any point  $p$  with radius 1:

$$B_1(p) = \{q : \tilde{d}(p, q) < 1\} = \{p\}, \quad \text{while} \quad \{q : \tilde{d}(p, q) \leq 1\} = \mathbb{R}^n.$$

Notice that the open ball is finite and hence closed. In particular, the closure of  $B_1(p)$  is just  $\{p\}$  and not  $\{q : \tilde{d}(p, q) \leq 1\}$ .

- 6) We need to show that  $K$  is compact or that every open cover of  $K$  contains a finite subcover. Let  $\{\mathcal{U}_\alpha\}_{\alpha \in A}$  be an open cover of  $K$ , so

$$K = \{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\} \subseteq \bigcup_{\alpha \in A} \mathcal{U}_\alpha \implies \exists \alpha_0 \in A \text{ such that } 0 \in \mathcal{U}_{\alpha_0}$$

Since  $\mathcal{U}_{\alpha_0}$  is open,  $\exists \varepsilon > 0$  with  $B_\varepsilon(0) \subseteq \mathcal{U}_{\alpha_0}$ . Because  $\varepsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that  $n > N \implies \frac{1}{N} < \varepsilon$ . Hence the open set  $\mathcal{U}_{\alpha_0}$  contains all of the  $\frac{1}{n}$  with  $n > N$ , i.e., it contains all but finitely many elements of  $K$ .

Now, for  $i = 1, 2, \dots, N$ ,  $\frac{1}{i} \in K$ . So  $\exists \alpha_i \in A$  such that  $\frac{1}{i} \in \mathcal{U}_{\alpha_i}$ . So we have shown that

$$K \subseteq \bigcup_{i=0}^N \mathcal{U}_{\alpha_i},$$

a finite subcover of  $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ . So every open cover of  $K$  contains a finite subcover, which shows that  $K$  is compact.

- 7) We have  $\{\mathcal{U}_\alpha\}_{\alpha \in A}$  an open cover of  $K$ . Define

$$\mathcal{V}_{\alpha, n} = \{x \in \mathcal{U}_\alpha \mid B_{\frac{1}{n}}(x) \subseteq \mathcal{U}_\alpha\}^\circ \text{ for all } \alpha \in A, n \in \mathbb{N}.$$

The  $\mathcal{U}_\alpha$  are open, so for any point  $x \in \mathcal{U}_\alpha$ , there is some  $n \in \mathbb{N}$  such that

$$B_{\frac{1}{n}}(x) \subseteq \mathcal{U}_\alpha \implies B_{\frac{1}{n}}(x) \subseteq \{y \in \mathcal{U}_\alpha \mid B_{\frac{1}{n}}(y) \subseteq \mathcal{U}_\alpha\} \implies x \in \mathcal{V}_{\alpha, n}. \text{ Hence } \bigcup_{n \in \mathbb{N}} \mathcal{V}_{\alpha, n} = \mathcal{U}_\alpha.$$

So taking the union over all  $\alpha \in A$ , we have

$$\bigcup_{\substack{\alpha \in A \\ n \in \mathbb{N}}} \mathcal{V}_{\alpha, n} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha \supseteq K.$$

So  $\{\mathcal{V}_{\alpha, n}\}_{\substack{\alpha \in A \\ n \in \mathbb{N}}}$  is an open cover of  $K$  (each set is an interior, thus open). By the compactness of  $K$ , there exists a finite subcover  $\{\mathcal{V}_{\alpha_i, n_i}\}_{i=1}^N$ . Let  $\delta = (\max_{1 \leq i \leq N} n_i)^{-1}$ . Then  $\forall x \in K, \exists i' \in \{1, 2, \dots, N\}$  with

$$x \in \mathcal{V}_{\alpha_{i'}, n_{i'}} \implies B_{\frac{1}{n_{i'}}}(x) \subseteq \mathcal{U}_{\alpha_{i'}}.$$

Since  $\delta^{-1} = \max_{1 \leq i \leq N} n_i \geq n_{i'}$ , we have  $\delta \leq \frac{1}{n_{i'}}$ , so  $B_\delta(x) \subseteq B_{\frac{1}{n_{i'}}}(x) \subseteq \mathcal{U}_{\alpha_{i'}}$ . Thus our  $\delta$  has the prescribed property.