

18.100B Problem Set 3 Solutions

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- 1) We begin by defining $d : V \times V \rightarrow \mathbb{R}$ such that $d(x, y) = \|x - y\|$. Now to show that this function satisfies the definition of a metric. $d(x, y) = \|x - y\| \geq 0$ and

$$d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$$

So the function is positive definite.

$$d(x, y) = \|x - y\| = \| -1(y - x) \| = | -1 | \|y - x\| = \|y - x\| = d(y, x)$$

Thus the function is symmetric. Finally,

$$d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$$

So the triangle inequality holds. Therefore d is a metric.

- 2) Once again we must verify the properties of a metric. We have defined d_1 as

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Since d is a metric, it only takes nonnegative values, so d_1 cannot be negative. $d_1(x, y)$ is zero exactly when $d(x, y)$ is, so only for $x = y$. Therefore d_1 is positive definite. Since d is symmetric, d_1 obviously inherits this property. Finally, for $x, y, z \in M$

$$\begin{aligned} d_1(x, y) + d_1(y, z) &= \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} = \frac{d(x, y) + d(y, z) + 2d(x, y)d(y, z)}{1 + d(x, y) + d(y, z) + d(x, y)d(y, z)} \\ &\geq \frac{d(x, y) + d(y, z) + d(x, y)d(y, z)}{1 + d(x, y) + d(y, z) + d(x, y)d(y, z)} = 1 - \frac{1}{1 + d(x, y) + d(y, z) + d(x, y)d(y, z)} \\ &\geq 1 - \frac{1}{1 + d(x, y) + d(y, z)} \geq 1 - \frac{1}{1 + d(x, z)} = \frac{d(x, z)}{1 + d(x, z)} = d_1(x, z) \end{aligned}$$

So the triangle inequality holds, thus we have a metric. It is easy to see that this metric never takes on a value larger than 1, since $d(x, y) < 1 + d(x, y)$, so under the metric d_1 , M is bounded.

- 3) a) $A, B \subseteq M$, M a metric space. Suppose $x \in A^\circ \cup B^\circ$. Without loss of generality, say $x \in A^\circ$. Therefore x is an interior point of A , so $\exists \epsilon > 0$ such that the ball of radius ϵ centered at x is contained in A , or $B_\epsilon(x) \subseteq A$. Since $A \subseteq A \cup B$,

$$B_\epsilon(x) \subseteq A \cup B \implies x \in (A \cup B)^\circ$$

This shows that $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.

- b) Now let $x \in A^\circ \cap B^\circ$. Therefore $x \in A^\circ$, so x is an interior point of A , hence $\exists \epsilon_1 > 0$ such that $B_{\epsilon_1}(x) \subseteq A$. Similarly, $x \in B^\circ \implies \exists \epsilon_2 > 0$ such that $B_{\epsilon_2}(x) \subseteq B$. Let $\delta = \min(\epsilon_1, \epsilon_2)$. By the triangle inequality,

$$\delta \leq \epsilon_i \implies B_\delta(x) \subseteq B_{\epsilon_i}(x) \implies B_\delta(x) \subseteq A \text{ and } B_\delta(x) \subseteq B.$$

Therefore $B_\delta(x) \subseteq A \cap B$, so x is an interior point of $A \cap B$. Hence $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$.

Let $x \in (A \cap B)^\circ$. So $\exists \varepsilon > 0$ with $B_\varepsilon(x) \subseteq A \cap B$. Therefore $B_\varepsilon(x) \subseteq A$ so $x \in A^\circ$, and similarly $x \in B^\circ$. So $x \in A^\circ \cap B^\circ$. Thus $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. So these two sets are equal.

Let $A = (-1, 0]$ and $B = [0, 1)$. Then 0 is an interior point of neither A nor B , so $0 \notin A^\circ \cup B^\circ$. But $A \cup B = (-1, 1)$, so $0 \in (A \cup B)^\circ$. Therefore in this instance the two sets are unequal.

- 4) a) If $x \in \partial A$ then every ball around x intersects A and A^c . Thus $x \in \overline{A}$ and x is a limit point of A^c or $x \in A^c$ and x is a limit point of A . Either way, $x \in \overline{A} \cap \overline{A^c}$, and hence $\partial A \subseteq \overline{A} \cap \overline{A^c}$.

Now let $x \in \overline{A} \cap \overline{A^c}$. Since $x \in \overline{A}$, either $x \in A$ or x is a limit point of A , and in both cases any open ball around x intersects A . Similarly, $x \in \overline{A^c}$ implies any open ball around x intersects A^c . Therefore $x \in \partial A$, so $\overline{A} \cap \overline{A^c} \subseteq \partial A$. So these two sets are equal.

- b) Let $p \in \partial A$. By a), $p \in \overline{A}$. Suppose $p \in A^\circ$ then $\exists \varepsilon > 0$ such that $B_\varepsilon(p) \subseteq A$. But this is an open ball centered at p which does not intersect A^c , so $p \notin \partial A$. This contradiction implies that $p \notin A^\circ$.

Now suppose $p \in \overline{A} \setminus A^\circ$. For any $\varepsilon > 0$, $p \in \overline{A}$ gives that $B_\varepsilon(x)$ intersects A , and $p \notin A^\circ$ implies that $B_\varepsilon(x) \not\subseteq A$, so $B_\varepsilon(x)$ intersects A^c . So $p \in \partial A$, and this shows that $\partial A = \overline{A} \setminus A^\circ$.

- c) By a), ∂A can be written as the intersection of two closed sets. Thus ∂A is closed.

- d) Suppose A is closed. Then $\overline{A} = A$, so by a)

$$\partial A = \overline{A} \cap \overline{A^c} = A \cap \overline{A^c} \subseteq A$$

Conversely, note that for any set B , if $x \notin B$ and $x \notin \partial B$, then there is a positive $r > 0$ such that $B_r(x) \subseteq B^c$ and hence $x \notin \overline{B}$. This implies that

$$\text{for any set } B, \overline{B} \subseteq B \cup \partial B.$$

So if $\partial A \subseteq A$, then $\overline{A} \subseteq A \cup \partial A = A \subseteq \overline{A}$ i.e., $A = \overline{A}$ hence A is closed.

- 5) We will show that $S_r(x) := \{y : d(x, y) = r\}$ is the boundary of $B_r(x)$. It will follow from the previous exercise that

$$\overline{B_r(x)} = \partial B_r(x) \cup B_r(x) = \{y : d(x, y) \leq r\}.$$

It is clear that if y is such that $d(x, y) = r$ then $y \in \partial B_r(x)$ since any ball around y will have points that are closer to x and points that are further away. We just have to show that if $d(x, y) \neq r$, then y is not in $\partial B_r(x)$.

But if $d(x, y) < r$ then for any $0 < \varepsilon < r - d(x, y)$ the ball of radius ε around y is all inside $B_r(x)$ and $y \notin \partial B_r(x)$; and if $d(x, y) > r$ then for any $0 < \delta < d(x, y) - r$ the ball of radius δ around y is all outside of $B_r(x)$ so that again $y \notin \partial B_r(x)$. Thus $\partial B_r(x)$ is precisely $S_r(x)$ and we are done.

Here is an example of a different metric space where this result is not true: Consider \mathbb{R}^n with the discrete metric,

$$\tilde{d}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

and the ball around any point p with radius 1:

$$B_1(p) = \{q : \tilde{d}(p, q) < 1\} = \{p\}, \quad \text{while} \quad \{q : \tilde{d}(p, q) \leq 1\} = \mathbb{R}^n.$$

Notice that the open ball is finite and hence closed. In particular, the closure of $B_1(p)$ is just $\{p\}$ and not $\{q : \tilde{d}(p, q) \leq 1\}$.

- 6) We need to show that K is compact or that every open cover of K contains a finite subcover. Let $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ be an open cover of K , so

$$K = \{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\} \subseteq \bigcup_{\alpha \in A} \mathcal{U}_\alpha \implies \exists \alpha_0 \in A \text{ such that } 0 \in \mathcal{U}_{\alpha_0}$$

Since \mathcal{U}_{α_0} is open, $\exists \varepsilon > 0$ with $B_\varepsilon(0) \subseteq \mathcal{U}_{\alpha_0}$. Because $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $n > N \implies \frac{1}{N} < \varepsilon$. Hence the open set \mathcal{U}_{α_0} contains all of the $\frac{1}{n}$ with $n > N$, i.e., it contains all but finitely many elements of K .

Now, for $i = 1, 2, \dots, N$, $\frac{1}{i} \in K$. So $\exists \alpha_i \in A$ such that $\frac{1}{i} \in \mathcal{U}_{\alpha_i}$. So we have shown that

$$K \subseteq \bigcup_{i=0}^N \mathcal{U}_{\alpha_i},$$

a finite subcover of $\{\mathcal{U}_\alpha\}_{\alpha \in A}$. So every open cover of K contains a finite subcover, which shows that K is compact.

- 7) We have $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ an open cover of K . Define

$$\mathcal{V}_{\alpha, n} = \{x \in \mathcal{U}_\alpha \mid B_{\frac{1}{n}}(x) \subseteq \mathcal{U}_\alpha\}^\circ \text{ for all } \alpha \in A, n \in \mathbb{N}.$$

The \mathcal{U}_α are open, so for any point $x \in \mathcal{U}_\alpha$, there is some $n \in \mathbb{N}$ such that

$$B_{\frac{1}{n}}(x) \subseteq \mathcal{U}_\alpha \implies B_{\frac{1}{n}}(x) \subseteq \{y \in \mathcal{U}_\alpha \mid B_{\frac{1}{n}}(y) \subseteq \mathcal{U}_\alpha\} \implies x \in \mathcal{V}_{\alpha, n}. \text{ Hence } \bigcup_{n \in \mathbb{N}} \mathcal{V}_{\alpha, n} = \mathcal{U}_\alpha.$$

So taking the union over all $\alpha \in A$, we have

$$\bigcup_{\substack{\alpha \in A \\ n \in \mathbb{N}}} \mathcal{V}_{\alpha, n} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha \supseteq K.$$

So $\{\mathcal{V}_{\alpha, n}\}_{\substack{\alpha \in A \\ n \in \mathbb{N}}}$ is an open cover of K (each set is an interior, thus open). By the compactness of K , there exists a finite subcover $\{\mathcal{V}_{\alpha_i, n_i}\}_{i=1}^N$. Let $\delta = (\max_{1 \leq i \leq N} n_i)^{-1}$. Then $\forall x \in K, \exists i' \in \{1, 2, \dots, N\}$ with

$$x \in \mathcal{V}_{\alpha_{i'}, n_{i'}} \implies B_{\frac{1}{n_{i'}}}(x) \subseteq \mathcal{U}_{\alpha_{i'}}.$$

Since $\delta^{-1} = \max_{1 \leq i \leq N} n_i \geq n_{i'}$, we have $\delta \leq \frac{1}{n_{i'}}$, so $B_\delta(x) \subseteq B_{\frac{1}{n_{i'}}}(x) \subseteq \mathcal{U}_{\alpha_{i'}}$. Thus our δ has the prescribed property.