

18.100B Problem Set 9 Solutions

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1) First we need to show that

$$f_n(x) = \frac{1}{nx + 1}$$

converges pointwise but not uniformly on $(0, 1)$. If we fix some $x \in (0, 1)$, we have that

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{1}{nx + 1} = 0.$$

Thus the f_n converge pointwise. However, consider $\varepsilon = \frac{1}{4}$, and let $N \in \mathbb{N}$. Then

$$f_N\left(\frac{1}{N}\right) = \frac{1}{N\left(\frac{1}{N} + 1\right)} = \frac{1}{2} \not\leq \varepsilon.$$

So the f_n do not converge to zero uniformly. Now we consider

$$g_n = \frac{x}{nx + 1}.$$

Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be larger than $\frac{1}{\varepsilon}$. Then for any $x \in (0, 1)$, $M \geq N$, we have

$$g_M(x) = \frac{x}{Mx + 1} = \frac{1}{M + \frac{1}{x}} < \frac{1}{M} \leq \frac{1}{N} < \varepsilon.$$

Thus, g_n converges to 0 uniformly on $(0, 1)$.

2) The functions f_n are defined on \mathbb{R} by

$$f_n = \frac{x}{1 + nx^2}.$$

First we show $f_n \rightarrow 0$. For any $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon^2}$. Then for $n > N$, if $|x| \leq \varepsilon$, $|x| < \varepsilon(1 + nx^2)$ so $|f_n(x)| < \varepsilon$. If $|x| > \varepsilon$,

$$|n\varepsilon x| > n\varepsilon^2 > 1 \implies |n\varepsilon x^2| > |x| \implies \varepsilon > \left| \frac{x}{nx^2} \right| > \left| \frac{x}{1 + nx^2} \right| = |f_n(x)|.$$

Thus f_n converges uniformly to $0 = f$.

Differentiating, we find

$$f'_n(x) = \frac{(1 + nx^2) \cdot 1 - x \cdot (2nx)}{(1 + nx^2)^2} = \frac{1 - nx^2}{(1 + nx^2)^2}.$$

Thus,

$$|f'_n(x)| = \left| \frac{1 - nx^2}{(1 + nx^2)^2} \right| = \frac{|1 - nx^2|}{(1 + nx^2)^2} \leq \frac{1 + nx^2}{(1 + nx^2)^2} = \frac{1}{1 + nx^2}.$$

So for all $x \neq 0$, $f'_n(x) \rightarrow 0$, but $f_n(0) = 1$, $\forall n \in \mathbb{N}$. So if $g(x) = \lim f_n(x)$, $g(0) = 1 \neq 0 = f'(0)$. But $f'(x) = 0$, and thus exists, for all x .

For all $x \neq 0$, $f'(x) = 0 = g(x)$. We showed above that $f_n \rightarrow f$ uniformly on all of \mathbb{R} . Finally, $f'_n \rightarrow g$ uniformly away from zero, that is, on any interval that does not have zero as a limit point. This is true because the denominator of f'_n can be made arbitrarily large compared to the numerator, for $|x| > \varepsilon$.

- 3) So we have $f_n \rightarrow f$ uniformly for f_n bounded. Then for $\varepsilon = 1$, $\exists N$ such that $\forall n > N$, $x \in E$, $|f_n(x) - f(x)| < 1$. Thus if f_n is bounded by B_n , $\forall x \in E$, $|f(x)| < 1 + B_N$ which implies that $\forall n > N$, $x \in E$, $f_n(x) < B_N + 2$. Thus (f_n) is uniformly bounded by $\max\{B_1, B_2, \dots, B_{N-1}, B_N + 2\}$.

If the f_n are converging pointwise, f need not be bounded. For example, on $(0, 1)$, if

$$f_n(x) = \frac{1}{x + \frac{1}{n}},$$

then $f_n \rightarrow \frac{1}{x}$ pointwise, and each f_n is bounded by n , but of course $\frac{1}{x}$ is unbounded on $(0, 1)$.

- 4) We have $f_n \rightarrow f$ and $g_n \rightarrow g$ uniformly.

- a) Given $\varepsilon > 0$, $\exists M, N \in \mathbb{N}$ such that for any $m > M$, $n > N$, $|f_m - f| < \frac{\varepsilon}{2}$ and $|g_n - g| < \frac{\varepsilon}{2}$. So for $n, m > \max\{M, N\}$

$$|(f_n + g_n) - (f + g)| = |(f_n - f) + (g_n - g)| \leq |f_n - f| + |g_n - g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $(f_n + g_n) \rightarrow (f + g)$ uniformly.

- b) If each f_n and g_n is bounded on E , by the previous problem they are uniformly bounded, say by A and B . Say without loss of generality $A \geq B$. Then for $\varepsilon > 0$, choose N such that if $n > N$, $|f_n - f| < \frac{\varepsilon}{2A}$ and $|g_n - g| < \frac{\varepsilon}{2A}$. Then

$$|f_n g_n - f g| = |f_n g_n - f g_n + f g_n - f g| \leq |f_n - f| |g_n| + |f| |g_n - g| < \frac{\varepsilon}{2A} A + \frac{\varepsilon}{2A} A = \varepsilon.$$

So $f_n g_n$ converges to $f g$ uniformly.

- 5) Define

$$f(x) = x, \quad g(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ q & \text{if, in lowest terms, } x = p/q \end{cases}$$

so that

$$f_n(x) = f(x) \left(1 + \frac{1}{n}\right), \quad \text{and} \quad g_n(x) = g(x) + \frac{1}{n}.$$

On any interval $[a, b]$, with $M = \max(|a|, |b|)$ we have

$$|f_n(x) - f(x)| = \frac{|x|}{n} \leq \frac{M}{n}, \quad |g_n(x) - g(x)| = \frac{1}{n}$$

so that $f_n \rightarrow f$ and $g_n \rightarrow g$ uniformly.

On the other hand, with $m = \min(|a|, |b|)$ we have

$$\begin{aligned} |f_n(x) g_n(x) - f(x) g(x)| &= |(f_n(x) - f(x)) g_n(x) + f(x) (g_n(x) - g(x))| \\ &= \frac{f(x)}{n} \left(g(x) + \frac{n+1}{n}\right) \geq \frac{m}{n} g(x). \end{aligned}$$

Now notice that if $L \in \mathbb{N}$ is larger than $b - a$ then there is an integer k such that $\frac{k}{L} \in [a, b]$, choosing $L \in \mathbb{N}$ larger than $\frac{n}{m}$ and larger than $b - a$ we get

$$\|f_n g_n - f g\| \geq \frac{m}{n} g\left(\frac{k}{L}\right) = \frac{m}{n} L > 1$$

and hence $f_n g_n$ does not converge to $f g$ uniformly.

6) We want to show $g \circ f_n \rightarrow g \circ f$ uniformly, for g continuous on $[-M, M]$. Since g is continuous on a compact set, it is uniformly continuous. So given $\varepsilon > 0$, $\exists \delta > 0$ such that if $|x - y| < \delta$, $|g(x) - g(y)| < \varepsilon$. Now since $f_n \rightarrow f$ uniformly, $\exists N \in \mathbb{N}$ such that for any $n > N$, $x \in E$, $|f_n(x) - f(x)| < \delta$. Thus $|g(f_n(x)) - g(f(x))| < \varepsilon$, so $|g \circ f_n(x) - g \circ f(x)| < \varepsilon$. Thus we have shown $(g \circ f_n) \rightarrow g \circ f$ uniformly on E .

7) a) First we claim that, for every $x \in [0, 1]$,

$$0 \leq P_n(x) \leq P_{n+1}(x) \leq \sqrt{x}.$$

This is clearly true for $n = 0$, so assume inductively that it is true for $n = k$ and notice that

$$\begin{aligned} \sqrt{x} - P_{k+1}(x) &= \sqrt{x} - \left[P_k(x) + \frac{1}{2}(x - P_k(x)^2) \right] \\ &= \sqrt{x} - P_k(x) - \frac{1}{2}(\sqrt{x} - P_k(x))(\sqrt{x} + P_k(x)) \\ &= (\sqrt{x} - P_k(x)) \left(1 - \frac{1}{2}(\sqrt{x} + P_k(x)) \right) \geq 0. \end{aligned}$$

It follows that $P_{k+1}(x) \leq \sqrt{x}$, and then

$$P_{k+1}(x) - P_k(x) = \frac{1}{2}(x - P_k(x)^2) \geq 0$$

shows that $P_k(x) \leq P_{k+1}(x)$, and proves the claim.

Notice that for every fixed x , the sequence $(P_n(x))$ is monotone increasing and bounded above (by \sqrt{x}), it follows that this sequence converges, to say $f(x)$. This function $f(x)$ is non-negative and satisfies

$$f(x) = \lim_{n \rightarrow \infty} P_{n+1}(x) = \lim_{n \rightarrow \infty} \left[P_n(x) + \frac{1}{2}(x - P_n(x)^2) \right] = f(x) + (x - f(x)^2)$$

which implies $f(x) = \sqrt{x}$, and hence the polynomials converge pointwise to \sqrt{x} on $[0, 1]$. Since they are continuous and converge monotonically to a continuous function on $[0, 1]$, a compact set, they converge uniformly (by Dini's theorem).

b) Here we use a something similar to problem 6 along with the above work to show that $P_n(x^2) \rightarrow |x|$ on $[-1, 1]$. The difference between this and problem 6 is we need $(f_n \circ g) \rightarrow (f \circ g)$, but this is easier. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ with $\forall x \in [0, 1]$, $n > N$, $|P_n(x) - \sqrt{x}| < \varepsilon$. So for all $x \in [-1, 1]$, $n > N$, $|P_n(x^2) - \sqrt{x^2}| < \varepsilon$. Since $|x| = \sqrt{x^2}$, we have show that the polynomials $P_n(x^2)$ converge uniformly to $|x|$ on $[-1, 1]$.