SOLUTIONS TO PS4

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Solution/Proof of Problem 1. *Consider the open set*

$$B_n = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1 - \frac{1}{n}\right\}.$$

Then we can see that $E \subset \cup B_n$ *because for any point* $(x, y) \in E$, $x^2 + y^2 < 1$, *we can find an* n *big enough such that* $x^2 + y^2 < 1 - \frac{1}{n}$, *i.e.* $(x, y) \in B_n$.

It is easy to see there is no finite subcover.

Solution/Proof of Problem 2. *At first, from the definition, we have* $d(x, x) = 0$. *From the equality*

$$d(x, y) = ||x|| + ||y|| = ||y|| + ||x|| = d(y, x),$$

we have $d(x, y) = d(y, x)$. *We also have*

$$d(x, y) = ||x|| + ||y|| = (\sum_{i=1}^{n} x_i^2)^{1/2} + (\sum_{i=1}^{n} y_i^2)^{1/2} \geq 0$$

and it is easy to see that $d(x, y) = 0$ *iff* $x = y = 0$.

From

$$||x|| + ||y|| \leq ||x|| + ||z|| + ||z|| + ||y||$$

we have $d(x, y) \leq d(x, z) + d(y, z)$.

So d *is a metric on* $\mathbb{R}^n$.

Open set in $(\mathbb{R}^k, d)$ may be not open in $(\mathbb{R}^k, d_{Euclid})$. For example, consider the open ball in $(\mathbb{R}^k, d)$,

$$B_r(x) = \{y \in \mathbb{R}^k : d(x, y) < r\}.$$

When $r > ||x||$, then we have

$$\begin{aligned} B_r(x) &= \{y \in \mathbb{R}^k : d(x, y) < r\} = \{y \in \mathbb{R}^k : ||x||| + ||y|| < r\} \\ &= \{y \in \mathbb{R}^k : ||y|| < r - ||x||\}, \end{aligned}$$

is just the open ball $B_{r-||x||}(0)$ under the Euclidean metric.

But when $r < ||x||$, then we have

$$B_r(x) = \{y \in \mathbb{R}^k : d(x, y) < r\} = \{y \in \mathbb{R}^k : |x||| + ||y|| < r\} = \{x\},$$

is not open under the Euclidean metric.

Conversely, consider an open set $U \subset (\mathbb{R}^k, d_{Euclid})$. Since for any point $x \neq 0$, under the new metric, $B_r(x) = \{x\} \subset U$ for any $r < ||x||$; if $x = 0$, then $B_r(0) \subset \mathbb{R}^k$ for some small r under the new metric. So we can always find an open neighborhood of x in the new metric that is contained in U. This means U is open in the new metric.

Solution/Proof of Problem 3. *First, recall that in any metric space a finite set is compact. We will show that for the discrete metric, these are the only compact sets.*

Notice that every subset $\{x\}$ *which contains only one point in* X *is an open subset. Indeed, if* $Y = \{x\}$*, then, for any* $r < 1$*, we have* $B_r(x) = \{x\}$ *hence* $B_r(x) \subset Y$*, i.e.,* Y *is open.*

Now suppose Y *is a compact subset of* X*. We can consider an open cover* $Y \subset \bigcup_{y\in Y} U_y$*, where* $U_y = \{y\}$*. Since* Y *is compact, this cover has a finite subcover. So* $Y \subset \bigcup_{\substack{\text{finitely many} \\ y\in Y}} U_y$*. Hence* Y *is a finite set.*

Solution/Proof of Problem 4. *From the definition of* E*, we can see that* $E \subset \{x \in \mathbb{Q}, -3 < x < 3\}$*. So it is bounded.*

E **is closed**. *Recall that, in any metric space, a set* E *is closed if and only if its complement is open. If* x *is any point whose square is less than* 2 *or greater than* 3 *then it is clear that there is a nieghborhood around* x *that does not intersect* E*. Indeed, take any such neighborhood in the real numbers and then intersect with the rational numbers. So the only problem would be at points whose square is exactly* 2 *or* 3*, but we know that are no such points within the rational numbers.*

E **is not compact**. *Consider the open cover*

$$U_n = \{x \in \mathbb{Q} : 2 + 1/n < x^2 < 3 - 1/n\}, n \in \mathbb{N}.$$

It is easy to see that it has no finite subcover.

E **is open**. *Given any point* $x \in E$ *there is a neighborhood of* x *within the real numbers of elements whose square is between* 2 *and* 3*, intersect this with the rational numbers to see that* E *is an open subset of* $\mathbb{Q}$*.*

Solution/Proof of Problem 5. *Suppose* X *and* Y *are two compact sets. If* $\{U_\alpha\}$ *is an open cover for* $X \cup Y$*, then it is also an open cover of* X*. Since* X *is compact, there is a finite subcover* $\{U_\beta\}_{\beta\in I} \subset \{U_\alpha\}$ *which still covers* X*. Similarly, we also have a finite subcover* $\{U_\beta\}_{\beta\in J} \subset \{U_\alpha\}$ *which covers* Y*. Putting these covers together,*

$$\{U_\beta\}_{\beta\in I\cup J} \subset \{U_\alpha\},$$

we get a finite subcover of $X \cup Y$*. So by the definition,* $X \cup Y$ *is compact.*

Since X *is a compact set in a metric space, it is closed. Hence* $X \cap Y$ *is the intersection of a closed set with a compact set. From Theorem 2.35's corollary, we can see that* $X \cap Y$ *is compact.*

Solution/Proof of Problem 6. *The set* $\{1\}$ *has no limit points because any neighborhood of this point has only one element* 1*.*

The statement can be proved as the follows. Let $\{x_k\}_{k=1}^{\infty}$ *be a convergent sequence in a metric space with infinitely many distinct elements. Suppose* $\lim_{n\to\infty} x_n = x_0$*. Then by the definition of the limits, for any neighborhood of* x_0*, there are at most finitely many points in the sequence outside the neighborhood. So we can always choose a point different from* x_0 *in any neighborhood which means* x_0 *is a limit point of the set.*

Solution/Proof of Problem 7. *Because they are countable, it is possible to put the rational numbers in* $[0,1]$ *in a sequence,* (p_n)*. We claim that every point* $x \in [0,1]$ *is a limit of a subsequence of* (p_n)*.*

We proved in class that between any two real numbers there is a rational number, it follows that between any two real numbers there are infinitely many rational numbers. This allows us to construct a subsequence of p_n *converging to* x *as follows. Assume for simplicity that* x *and* $x + \frac{1}{10}$ *are both in* $[0,1]$*. From among the infinitely many rational numbers between* x *and* $x + \frac{1}{10}$*, let* p_{n_1} *be the first of the* p_n *to fall in* $x < p_{n_1} \leq x + \frac{1}{10}$*. Because there are inifinitely many rational numbers between* x *and* p_{n_1} *we can pick* p_{n_2} *to be the first rational number in the sequence* p_n *ocurring after* p_{n_1} *to fall inside* $x < p_{n_2} \leq x + \frac{1}{10^2}$*. Similarly, we choose* p_{n_3} *to be the first rational number in the sequence* p_n *ocurring after* p_{n_2} *to fall inside* $x < p_{n_3} \leq x + \frac{1}{10^3}$*. Continuing in this fashion, we achieve a subsequence of* p_n*, which we denote* p_{n_k} *with the property that*

$$x < p_{n_k} \leq x + \frac{1}{10^k}, \text{ for any } k \in \mathbb{N}.$$

Hence $p_{n_k} \to x$*.*

This was done under the assumption that both x *and* $x + \frac{1}{10}$ *were both in* $[0,1]$*. If that is not the case, but* $x < 1$ *then we can find* N *such that* x *and* $x + \frac{1}{10^N}$ *are both in* $[0,1]$ *and we can start the construction from there. Finally, if* $x = 1$ *then* x *and* $x - \frac{1}{10}$ *are both in* $[0,1]$ *and we can carry out the above construction requiring that* p_{n_k} *satisfy* $x - \frac{1}{10^k} \leq p_{n_k} < x$ *for every* k*.*

Hence in every case we obtain a subsequence of (p_n) *that converges to* x*.*

Solution/Proof of Problem 8. *The question is asking: If* A *is connected, does the interior of* A *have to be connected? does the closure of* A *have to be connected?*

Closure of a connected set is always connected. Suppose $\overline{E} = A \cup B$*, where* $A \cap \overline{B} = \emptyset$ *and* $\overline{A} \cap B = \emptyset$*, we show that* $\overline{E}$ *is connected by proving that either* A *or* B *must be empty.*

We know that E *is connected and* $E = (A \cap E) \cup (B \cap E)$ *with* $A \cap E$*,* $B \cap E$ *separated sets, hence we must have* $A \cap E = \emptyset$ *or* $B \cap E = \emptyset$*. Say that* $A \cap E = \emptyset$*, then* $E \subseteq B$ *and hence* $\overline{E} \subseteq \overline{B}$*. But we know that* $A \cap \overline{B} = \emptyset$*, hence*

$$A = A \cap (A \cup B) = A \cap \overline{E} \subseteq A \cap \overline{B} = \emptyset,$$

which implies that $\overline{E}$ *is connected.*

The interior of a connected set may not be connected. Consider two tangent closed disk in $\mathbb{R}^2$*:* $\overline{B_1((0,1))}$ *and* $\overline{B_1((0,-1))}$*. The union will give us a connected set. But the interior part of it will be two separated open balls.*