

# 18.100B Practice for the first midterm

## Solutions.

### Problems.

- 1) Let  $(\mathcal{M}, d)$  be an arbitrary metric space.  
a) State the definition of a connected subset of  $\mathcal{M}$ .

**Solution.** See Definition 2.45 in Rudin.

- b) Prove that  $E \subseteq M$  is connected if and only if every non-empty proper subset has a non-empty boundary in  $E$ .

**Solution.** Notice that the equivalent statement

$E$  is separated if and only if there is a proper non-empty subset with empty boundary in  $E$ ,

follows from the fact that  $A \cup B$  is a separation of  $E$  if and only if  $A$  and  $B = E \cap A^c$  have no boundary in  $E$ .

- 2) Let  $(\mathcal{M}, d)$  be an arbitrary metric space (e.g., not necessarily Euclidean space).  
a) Show that a compact subset of  $\mathcal{M}$  is necessarily closed and bounded.

**Solution.** See Theorem 2.34 in Rudin for a proof that compact sets are closed. To prove that a compact set  $K \subseteq \mathcal{M}$  is bounded, pick any point  $p \in \mathcal{M}$  and consider the open sets  $B_n(p)$ . These cover  $K$  (indeed, they cover  $\mathcal{M}$ ), hence there is a finite subcover of  $K$  and hence  $K$  is contained in  $B_N(p)$  for large enough  $N$ , i.e.,  $K$  is bounded.

- b) Give an example of a metric space with a closed and bounded subset that is *NOT* compact.

*Hint:* Use the discrete metric  $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

**Solution.** Notice that any subset of a metric space with the discrete metric is closed and bounded. However, only finite subsets are compact (by a homework question), hence any infinite subset is closed, bounded, and not compact.

- 3) Show that  $\sqrt{2} + \sqrt{3}$  is irrational.

*Hint:* Show that  $\sqrt{2} + \sqrt{3} \in \mathbb{Q} \implies \sqrt{2} \in \mathbb{Q}$ .

**Solution.** Let  $\sqrt{2} + \sqrt{3} = r$  then  $\sqrt{3} = r - \sqrt{2}$  and squaring both sides we get  $3 = r^2 - 2\sqrt{2} + 2$ . If  $r$  is rational, then solving this equation for  $\sqrt{2}$  would give a rational expression for  $\sqrt{2}$  which we know does not exist.

- 4) Let  $(\mathcal{M}, d)$  be an arbitrary metric space (e.g.,  $\mathcal{M}$  is not necessarily complete). If  $(x_n)$  and  $(y_n)$  are both Cauchy sequences and  $d_n = d(x_n, y_n)$ , show that  $(d_n)$  is a convergent sequence of real numbers.

**Solution.** Because  $\mathbb{R}$  is complete, we only need to show that  $d_n$  is Cauchy. Repeated use of the

triangle inequality shows that

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \implies d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_n, y_m)$$

and since the same is true reversing the roles of  $m$  and  $n$ , we find

$$|d_n - d_m| = |d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m).$$

Thus  $(x_n)$  Cauchy and  $(y_n)$  Cauchy together imply  $(d_n)$  Cauchy and hence convergent.

- 5) Let  $(\mathcal{M}, d)$  be an arbitrary metric space. If  $G \subseteq \mathcal{M}$  is open, and  $A$  is any subset of  $\mathcal{M}$ , show that

$$G \cap A = \emptyset \iff G \cap \bar{A} = \emptyset$$

**Solution.** Clearly  $G \cap \bar{A} = \emptyset$  implies  $G \cap A = \emptyset$ , so suppose  $G \cap A = \emptyset$ , we need to show that no point of  $G$  is a limit point of  $A$ . But if  $x \in G$  then, because  $G$  is open, there is an open ball around  $x$  that stays in  $G$  and hence does not intersect  $A$ , which implies that  $x$  is not a limit point of  $A$ .

- 6) Show that if  $x, y \in \mathbb{R}$  and  $x < y$  then there exists an *irrational* number between  $x$  and  $y$ . (You may use the existence of a rational number between  $x$  and  $y$ .)

**Solution.** Let  $r$  be a rational number satisfying  $x < r < y$ , we can find a large enough  $N$  so that

$$x < r + \frac{\sqrt{2}}{N} < y \quad \left( \iff N(y - r) > \sqrt{2} \right)$$

and notice that if  $r + \frac{\sqrt{2}}{N} = q$  were rational then  $\sqrt{2} = N(q - r)$  would be rational.

An alternate proof is to note that there are uncountably many reals between  $x$  and  $y$  and there are only countably many rationals, so there must be irrationals between  $x$  and  $y$ .