# 18.100B Problem Set 7 Solutions

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1) We have $a_i > 0$ and $a_{i+1} \leq a_i$ for all $i = 0, 1, 2, ...$, and $\lim_{i \to \infty} a_i = 0$, and we want to show the convergence of

$$\sum_{i=0}^{\infty} (-1)^i a_i = a_0 - a_1 + a_2 - ...$$

So we define $s_n$ to be the partial sums of the first $n + 1$ terms of the sum:

$$s_n = \sum_{i=0}^{n} (-1)^i a_i$$

So for any $k \in \mathbb{N}$, we have $s_{2k} - s_{2k-2} = (-1)^{2k-1} a_{2k-1} + (-1)^{2k} a_{2k} = a_{2k} - a_{2k-1} \leq 0$ by the monotonicity of $a_n$. Therefore $s_{2k} \leq s_{2k-2}$, so $s_{2k}$ is decreasing. Similarly, $s_{2k-1}$ is increasing. Also, $s_{2k} - s_{2k-1} = (-1)^{2k} a_{2k} = a_{2k} > 0$, so $s_{2k} > s_{2k-1}$. These combine to give $s_{2k-1} < s_{2k'}$ for any $k, k' = 0, 1, 2, ...$, since choosing $N > \max\{k, k'\}$, we have

$$s_{2k-1} \leq s_{2N-1} < s_{2N} \leq s_{2k'}.$$

So $s_{2k}$ and $s_{2k-1}$ are both monotonic and bounded, so they each converge. However, $s_{2k} - s_{2k-1} = a_{2k} \longrightarrow 0$, so they must converge to the same limit. Therefore $s_n$ converges, so the sum is convergent.

2) Our function $f$ is defined on $(0, 1)$ by

$$f = \begin{cases} \frac{1}{q} & \text{if, in lowest terms, } x = \frac{p}{q} \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

First we show $f$ is discontinuous at every rational. For $r = \frac{p}{q} \in \mathbb{Q}$ (in lowest terms), let $\varepsilon = \frac{1}{2q}$. Then for any $\delta > 0$, $\exists x \notin \mathbb{Q}$ with $r < x < r + \delta$, so $|f(r) - f(x)| = |\frac{1}{q} - 0| = \frac{1}{q} > \frac{1}{2q} = \varepsilon$. This proves discontinuity at $r$.

Now we want to show continuity at irrationals, so let $x \notin \mathbb{Q}$, $0 < x < 1$. Given $\varepsilon > 0$, we need to find a $\delta > 0$ such that for every $y$ with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$. Since $f(x) = 0$ and $f(y) \geq 0$, we need $f(y) < \varepsilon$. But $\varepsilon > 0$ means we can find $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$. So $|f(y)| > \varepsilon$ means $y$, when written in lowest terms, has denominator smaller than $N$. But there are only finitely many fractions with denominator less than $N$ between 0 and 1, so for some $M$, $\{y_i\}_{i=1}^{M} = \{y \in (0, 1) | f(y) > \varepsilon\}$. So we let $\delta = \min_{1 \leq i \leq M} |x - y_i|$, which exists and is greater than 0 since there are only finitely many $y_i$. Then if $y \in (0, 1)$ is such that $|x - y| < \delta$, $y \neq y_i$ $\forall i \in \{1, 2, ..., M\}$, so $f(y) < \varepsilon$. Therefore $f$ is continuous at $x$.

3) We have $f, g : \mathcal{M} \longrightarrow \mathcal{N}$, and $\mathcal{Q} \subseteq \mathcal{M}$ is dense.

a) We need to show $f(\mathcal{Q})$ is dense in $f(\mathcal{M})$. So let $K \subseteq \mathcal{N}$ be closed, with $f(\mathcal{Q}) \subseteq K$. Then by continuity of $f$, $f^{-1}(K)$ is closed, and $f^{-1}(K)$ contains $f^{-1}(f(\mathcal{Q})) \supseteq \mathcal{Q}$. Since $\mathcal{Q}$ is dense in $\mathcal{M}$, $f^{-1}(K) = \mathcal{M}$. Hence $f(\mathcal{M}) \subseteq K$, so $f(\mathcal{Q})$ is dense in $f(\mathcal{M})$.

b) Now we have $f = g$ on $\mathcal{Q}$. Now consider the function $\phi : \mathcal{M} \longrightarrow \mathbb{R}$, with

$$\phi(x) = d_{\mathcal{N}}(f(x), g(x))$$

for $d_{\mathcal{N}}$ the distance function on $\mathcal{N}$. Since $d_{\mathcal{N}}$, $f$, and $g$ are all continuous, $\phi$ is also continuous. Therefore $\phi^{-1}(0)$ is a closed set in $\mathcal{M}$. But since $f = g$ on $\mathcal{Q}$,

$$\forall x \in \mathcal{Q}, f(x) = g(x) \Longrightarrow \phi(x) = d_{\mathcal{N}}(f(x), g(x)) = 0.$$

Thus $\mathcal{Q} \subseteq \phi^{-1}(0)$, which is closed, so by density,

$$\phi^{-1}(0) = \mathcal{M} \Longrightarrow \forall x \in \mathcal{M},\ 0 = \phi(x) = d_{\mathcal{N}}(f(x), g(x)) \Longrightarrow f(x) = g(x).$$

4) a) So we must find a continuous $f : E \longrightarrow \mathbb{R}$ with $E \subseteq \mathbb{R}$ bounded and $f(E)$ unbounded. Let

$$E = (0,1),\ f(x) = \frac{1}{x}$$

So $E$ is clearly bounded, $f(E) = (1, \infty)$ is unbounded, and $f$ is continuous: at $x \in (0,1)$, given $\varepsilon > 0$, let $\delta = \min\{\frac{x}{2}, \frac{1}{3}x^2\varepsilon\} > 0$. Then

$$f(x-\delta) - f(x) = \frac{1}{x-\delta} - \frac{1}{x} = \frac{\delta}{x(x-\delta)} \leq \frac{\frac{1}{3}x^2\varepsilon}{x(\frac{x}{2})} = \frac{2}{3}\varepsilon < \varepsilon.$$

And similarly, $f(x) - f(x+\delta) < \varepsilon$. Since $f$ is monotonically decreasing, this shows $f$ is continuous.

b) Now we have that $f$ is uniformly continuous, and $E$ is bounded. So for $\varepsilon = 1$, $\exists \delta > 0$ such that $\forall x, y \in E$, if $|x - y| < \delta$, $|f(x) - f(y)| < 1$. $E$ is bounded, so let $B \in \mathbb{N}$ such that $E \subseteq [-B, B]$. Then we can divide $[-B, B]$ into $\lceil \frac{4B}{\delta} \rceil$ closed intervals of length $\frac{\delta}{2}$, say $I_i$ for $i = 1, 2, ..., M$. Then choose $x_i \in I_i \cap E$, when $I_i \cap E \neq \emptyset$. Let $C = \max_i\{|f(x_i)|\} + 1$. Then for any $x \in E$, $x \in I_i$ for some $i$, so $|x - x_i| \leq \frac{\delta}{2} < \delta$. Thus, $|f(x) - f(x_i)| < 1$, so $|f(x)| < 1 + |f(x_i)| \leq C$. So $C$ bounds $f(E)$.

c) This is the easiest one: let $E = \mathbb{R}$, and $f(x) = x$. Then $f$ is uniformly continuous by choosing $\delta = \varepsilon$, since $|f(x) - f(y)| = |x - y|$, but $E = f(E) = \mathbb{R}$ are both unbounded.

5) $f : \mathcal{M} \longrightarrow \mathcal{N}$ is a uniformly continuous map between metric spaces.

a) We need to show that $f$ preserves Cauchy sequences. So we are given that $(x_n)$ is Cauchy. To show $(f(x_n))$ is Cauchy, let $\varepsilon > 0$. Then by uniform continuity, $\exists \delta > 0$ such that if $x, y \in \mathcal{M}$ with $d_{\mathcal{M}}(x, y) < \delta$, then $d_{\mathcal{N}}(f(x), f(y)) < \varepsilon$. Since $(x_n)$ is Cauchy, $\exists N \in \mathbb{N}$ such that $\forall n, m > N$, $d_{\mathcal{M}}(x_n, x_m) < \delta$. Therefore, $d_{\mathcal{N}}(f(x_n), f(x_m)) < \varepsilon$, for all $n, m > N$. So $(f(x_n))$ is Cauchy.

b) We have $g(x) = x^2$ on $\mathbb{R}$. $g$ is continuous, so given a Cauchy sequence $(x_n)$ in $\mathbb{R}$, it converges by the completeness of $\mathbb{R}$, so $(g(x_n))$ is convergent by the continuity of $g$, so it is Cauchy. But

$g$ is not uniformly continuous: for any $\delta > 0$, letting $x = \dfrac{1}{\delta}$ we have

$$g(x + \frac{\delta}{2})^2 - g(x) = (x^2 + 1 + \frac{\delta^2}{4}) - x^2 > 1.$$

So for $\varepsilon = 1$, no $\delta$ exists that satisfies uniform continuity on all of $\mathbb{R}$.

6) We have defined

$$d_E(x) = \inf_{z \in E} d(x, z),$$

and we want to show that

$$|d_E(x) - d_E(y)| \leq d(x, y).$$

So fix $\varepsilon > 0$. Then $\exists z_0 \in E$ with $d(x, z) < d_E(x) + \varepsilon$. Then $d(y, z) \leq d(x, z) + d(x, y) < d_E(x) + \varepsilon + d(x, y)$ by the triangle inequality. Therefore,

$$d_E(y) = \inf_{z \in E} d(y, z) \leq d_E(x) + \varepsilon + d(x, y) \Longrightarrow d_E(y) - d_E(x) \leq d(x, y) + \varepsilon$$

for any $\varepsilon > 0$. So $d_E(y) - d_E(x) \leq d(x, y)$. By symmetry, $d_E(x) - d_E(y) \leq d(x, y)$, so

$$|d_E(x) - d_E(y)| \leq d(x, y).$$

Uniform continuity follows immediately by letting $\delta = \varepsilon$.

7) So now we have $K, F \subseteq \mathcal{M}$, with $K \cap F = \emptyset$, $F$ closed and $K$ compact. By the previous exercise, $d_F$ is uniformly continuous, and thus continuous, positive function on $K$. Therefore $d_F$ attains its minimum on $K$, by compactness. So $\exists x \in K$ with $d_F(x) = \inf_{y \in K} d_F(y)$. Suppose this infimum is 0. Thus $d_F(x) = 0$, so $x$ is a limit point of $F$. But $F$ is closed, so $x \in F \Longrightarrow x \in K \cap F = \emptyset$, a contradiction. Therefore $\inf_{y \in K} d_F(y) = \delta > 0$. Therefore $\forall p \in K, q \in F$, $d(p, q) > \frac{\delta}{2} > 0$.

This doesn't hold for arbitrary $K, F \subseteq \mathcal{M}$ closed. For a counterexample, take $\mathcal{M} = \mathbb{R}^2$ and look at

$$K = \{(x, y) \in \mathbb{R}^2 | y = 0\},\ F = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq \frac{1}{x}\}$$

Both are closed, and for any $\delta > 0$, for $N > \dfrac{1}{\delta}$, $(N, 0) \in K$ and $(N, \dfrac{1}{N}) \in F$, and the distance betweent these is $\dfrac{1}{N} < \delta$.