

SOLUTIONS TO PS 10
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Solution/Proof of Problem 1. From the definition, we can find that

$$f_n(t) = \left(\frac{2}{3}\right)^n f_0(t) + \sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k.$$

Notice that $\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k = \frac{1}{1-2/3} = 3$ and since $|f_0(t)| = |\sin t| \leq 1$ we have

$$|f_n - 3| \leq \left|\left(\frac{2}{3}\right)^n\right| + \left|\sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k - 3\right| \text{ so we have } \forall \epsilon > 0,$$

- $\exists N_1$ s.t. $\forall n > N_1, \left(\frac{2}{3}\right)^n < \frac{\epsilon}{3}$;
- $\exists N_2$ s.t. $\forall n > N_1, \left|\sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k - 3\right| < \frac{\epsilon}{3}$.

So take $N = \max\{N_1, N_2\}$, and we have $\forall n > N$,

$$|f_n - 3| \leq \left|\left(\frac{2}{3}\right)^n\right| + \left|\sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k - 3\right| \leq \frac{2\epsilon}{3} < \epsilon.$$

So $f_n \rightarrow 3$ uniformly on \mathbb{R} .

In general, since $f_n(x) = T^n(f_0(x))$, T is a contraction, and the fixed point of T is 3, we always have pointwise convergence of f_n to 3. However, from the argument above we see that this is uniform convergence if and only if the function f_0 is bounded. Thus for $f_0(t) = t^2$, the convergence is uniform on any bounded subset of \mathbb{R} , but not on all of \mathbb{R} .

Solution/Proof of Problem 2. Since $t \geq 0$, we have $0 \leq \phi(t) \leq \frac{t}{2+t} \leq \frac{t}{2}$. So we have

$$0 \leq f_n(t) = \phi(f_{n-1}(t)) \leq \frac{1}{2} f_{n-1}(t) \leq \dots \leq \frac{1}{2^n} f_0(t) = \frac{1}{2^n} \phi(t) \leq \frac{1}{2^n} \frac{t}{2+t} \leq \frac{1}{2^n}.$$

From the convergence of $\sum \frac{1}{2^n}$, we have $\sum_{n=0}^K f_n(t) \rightarrow F(t)$ uniformly, since each partial sum $\sum_{n=0}^K f_n(t)$ is continuous, this implies that F is continuous.

Solution/Proof of Problem 3. Since differentiability is a local property, we only need to prove that f is differentiable on every subset $(-s, s) \subset \mathbb{R}$.

We have

$$\frac{d}{dt} \sin^2\left(\frac{t}{k}\right) = \frac{2}{k} \sin\left(\frac{t}{k}\right) \cos\left(\frac{t}{k}\right) = \frac{1}{k} \sin\left(\frac{2t}{k}\right),$$

so if $F_n(t) = \sum_{k=1}^n \sin^2\left(\frac{t}{k}\right)$, then

$$\frac{d}{dt} F_n = \sum_{k=1}^n \frac{1}{k} \sin\left(\frac{2t}{k}\right).$$

We can use $|\sin x| \leq |x|$ to see that $F'_n(t)$ is uniformly Cauchy; indeed, whenever $n < m$ we have

$$\begin{aligned} \|F'_n - F'_m\| &= \sup_{t \in [-s, s]} \left| \sum_{k=n}^m \frac{1}{k} \sin\left(\frac{2t}{k}\right) \right| \leq \sup_{t \in [-s, s]} \sum_{k=n}^m \left| \frac{1}{k} \sin\left(\frac{2t}{k}\right) \right| \\ &\leq \sup_{t \in [-s, s]} \sum_{k=n}^m \frac{1}{k} \left(\frac{2|t|}{k} \right) = 2s \sum_{k=n}^m \frac{1}{k^2} \end{aligned}$$

and since $\sum \frac{1}{k^2}$ converges, we can make this last sum as small as we like. It follows that $F'_n(t)$ converges uniformly, it's also clear that $F_n(0) \rightarrow 0$. From Theorem 7.17, we know that $F_n(t)$ converges to a function $F(t)$ such that $F'(t)$ exists and $F'(t) = \lim_{n \rightarrow \infty} F'_n(t)$. So we get the conclusion.

Solution/Proof of Problem 4. Since $f_n \rightarrow f$ uniformly, and f_n are continuous, so is f . Now for any $\epsilon > 0$, we have

- $\exists N_1$, s.t. $\forall n > N_1$, and $\forall x \in E$, $|f(x) - f_n(x)| < \frac{\epsilon}{3}$;
- $\exists \delta > 0$, s.t. $\forall |x - y| < \delta$, $|f(y) - f(x)| < \frac{\epsilon}{3}$;
- $\exists N_2$, s.t. $\forall n > N_2$, $|x - x_n| < \delta$.

So we have for $n > \max\{N_1, N_2\}$,

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \leq \frac{2\epsilon}{3} \leq \epsilon.$$

So we get the conclusion.

The converse can be formulated different ways. Here's one that's true: If (f_n) is a sequence of continuous functions that converge pointwise to a function f on a compact set E , and $\lim_{y \rightarrow x} f(y)$ always exists, then

$$f_n \rightarrow f \text{ uniformly} \iff \lim_{n \rightarrow \infty} f_n(x_n) = \lim_{n \rightarrow \infty} f(x_n) \text{ whenever } (x_n) \text{ converges.}$$

The proof of \rightarrow is above, to prove \leftarrow assume that f_n does not converge to f uniformly. This implies that

for some $\epsilon_0 > 0$ and for every $N \in \mathbb{N}$

there exists $n > N$ such that $\|f_n(x_n) - f(x_n)\| > \epsilon_0$

\iff for some $\epsilon_0 > 0$ and for every $N \in \mathbb{N}$

there exists $n > N$ and $y_n \in E$ such that $|f_n(y_n) - f(y_n)| > \epsilon_0$

Since E is compact, the sequence (y_n) has a convergent subsequence, which we denote (x_n) . Say that $\lim_{n \rightarrow \infty} f(x_n) = L$ and find $N' \in \mathbb{N}$ such that $n > N'$ implies $|f(x_n) - L| < \epsilon_0/2$. Then, for any $n > N'$ we have $|f_n(x_n) - L| > \epsilon_0/2$ and hence

$$\lim_{n \rightarrow \infty} f_n(x_n) \neq \lim_{n \rightarrow \infty} f(x_n),$$

which proves the converse.

Notice that if we do not require the original sequence to be continuous, then the converse is not true. Take $E = \mathbb{R}$. Consider a sequence of functions

$$f_n(x) = \begin{cases} 0 & x \in (-n, n) \\ 1 & \text{otherwise} \end{cases}$$

Then f_n converge to 0 pointwise and f_n does not converge uniformly to 0. But we can easily see that for any convergent sequence $\{x_n\}$, $f_n(x_n) \rightarrow 0$.

Solution/Proof of Problem 5. Form condition (b), we have

$$0 \leq \int_0^\infty f_n(t)dt \leq \int_0^\infty e^{-t} = 1.$$

So we have

$$0 \leq \lim_{T \rightarrow \infty} \int_T^\infty f_n(t)dt \leq \lim_{T \rightarrow \infty} \int_T^\infty e^{-t} = 0.$$

So for $\frac{\varepsilon}{3} > 0$, $\exists S$ s.t. $\forall n$

$$0 \leq \int_S^\infty f_n(t)dt \leq \int_S^\infty e^{-t} \leq \frac{\varepsilon}{3}.$$

On the other hand, from condition (a), we have for $\frac{\varepsilon}{3S} > 0$, $\exists N$ s.t. $\forall n > N$

$$\int_0^S f_n(t)dt \leq \int_0^S \frac{\varepsilon}{3S} dt = \frac{\varepsilon}{3}.$$

So we have for $\varepsilon > 0$, $\exists N$ s.t. $\forall n > N$

$$\int_0^\infty f_n(t)dt \leq \int_0^S f_n(t)dt + \int_S^\infty f_n(t)dt \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.$$

So $\lim_{n \rightarrow \infty} \int_0^\infty f_n(t)dt = 0$.

Condition (b) is necessary. In fact, we can consider $f_n(t) = \frac{t}{n}$, which satisfies condition (a). It is clear that the conclusion does not hold.

Solution/Proof of Problem 6. Since $\{f_n\}$ is equicontinuous, so for any $\frac{\varepsilon}{3} > 0$, $\exists \delta > 0$ s.t. for any two points $x, y \in K$, if $|x - y| < \delta$, then $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$.

Now consider an open covering $K = \bigcup_{x \in K} D_\delta(x)$ where $D_\delta(x)$ is a disc with center x and radius δ . Since K is compact, we can find finite disc to cover K . Let $K = \bigcup_{i=1}^n D_\delta(x_i)$,

For any $x \in K$, we have $x \in D_\delta(x_i)$ for some x_i . So $|f_n(x) - f_n(x_i)| < \frac{\varepsilon}{3}$.

For each i , $f_n(x_i) \rightarrow f(x_i)$, we have for $\frac{\varepsilon}{3} > 0$, $\exists N_i > 0$, s.t. $\forall n > N_i$, $|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3}$. Let $N = \max_i N_i$, so $\forall n > N$, $|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3}$ for any i .

So $\forall m, n > N$

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_m(x_i)| + |f_m(x_i) - f_n(x_i)| + |f_n(x_i) - f_n(x)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon.$$

So $f_n \rightarrow f$ uniformly.

Solution/Proof of Problem 7. Since f'_n is uniformly bounded, so there exists M s.t. $|f'_n(x)| \leq M, \forall x, n$.

For any $x \leq y$, by MVT, we have $\exists \xi \in [x, y]$ s.t.

$$|f_n(x) - f_n(y)| = |f'_n(\xi)(x - y)| \leq M|x - y|.$$

So for any $\varepsilon > 0$, $\exists \delta = \varepsilon/M > 0$ s.t. $\forall x, y, |x - y| < \delta$, $|f_n(x) - f_n(y)| < \varepsilon$.

So f_n is equicontinuous. Then from Arzela-Ascoli theorem, we got the conclusion.