18.100B, FALL 2002
PRACTICE TEST 1 WITH SOLUTIONS

Try each of the questions; they will be given equal value. You may use theorems from class, or the book, provided you can recall them correctly!

Problem 1

Consider the set S defined as follows. The elements of S are sequences, $\{s_n\}_{n=1}^{\infty}$ with all entries either 1 or 2 and with the additional property that every 2 is followed by a 1. Said more precisely, for every n, $s_n = 1$ or $s_n = 2$ and if $s_n = 2$ then $s_{n+1} = 1$. Say why precisely one of the following is true

(a) S is finite
(b) S is countably infinite
(c) S is uncountably infinite

and then decide which one is true and *prove* it.

Solution and remarks: By definition a set S is finite if it is either empty of else is in 1-1 correspondence with the set $\{1, \dots, n\}$ for some n. It is countably infinite if it is in 1-1 correspondence with $\mathbb{N}$ and it is uncountably infinite if it is neither finite nor countably infinite. Only one of these can hold (since we know that $\mathbb{N}$ cannot be in 1-1 correspondence with $\{1, \dots, n\}$ for any finite n.)

The set S is uncountably infinite. Here is a proof that reduces it to the case we looked at in class. Namely, we know that S' which consists of the set of sequences with values 0 or 1 is uncountably infinite. We show that S and S' are in 1-1 correspondence (and hence have the same cardinality by definition). Take a sequence in S' and replace evey occurence of 0 by two terms, 2 followed by 1. This gives a sequence in S. Moreover no two sequences in S' are mapped to the same sequence in S. Thus the map is injective. We can construct an inverse the same way, replace every pair $2, 1$ by one element 0. Thus S is indeed uncountably infinite.

It is also fairly straightforward to use the diagonalization procedure, but not completely trivial since you have to make sure that the new sequence is in S and different from the others.

Problem 2

Consider the metric space $M = [0, 1] = \{x \in \mathbb{R}; 0 \leq x \leq 1\}$ with the usual metric, $d(x, y) = |x - y|$. Is the set $A = [0, \frac{1}{2}) = \{x \in \mathbb{R}; 0 \leq x < \frac{1}{2}\}$ open as a subset of M? What is the closure of A as a subset of M? Is A compact? Is the closure of A compact? In each case justify your answer.

Solution and remarks: Everything is relative to the metric space $M = [0, 1]$.

(1) As a subset of M, A is indeed open. If $x \in A$ then $B(x, \epsilon) \subset A$ if $\epsilon = \frac{1}{2} - x$, since $|y - x| < \epsilon$, $y \in [0, 1]$ implies $y < \frac{1}{2}$ and hence $y \in A$.
(2) Clearly $\frac{1}{2}$ is a limit point of A so the closure $\bar{A} \supset [0, \frac{1}{2}]$. By the same argument as above $(\frac{1}{2}, 1]$ is open in $[0, 1]$ so this set is closed and hence $\bar{A} = [0, \frac{1}{2}]$.

(3) Since A is not closed it cannot be compact.
(4) By the Heine-Borel theorem $\bar{A} = [0, \frac{1}{2}]$ is compact since it is closed and bounded.

Usual errors with this sort of question are to say that A is not open, thinking of it as a subset of $\mathbb{R}$ this is certainly true but it is open as a subset of M. Similarly in the third part it does not follow directly from the fact that A is open that it is not compact! It does follow from the fact that it is not closed, but in a finite metric space (which this is not) there are open compact sets so something else has to be said.

Problem 3

Let M be a *compact* metric space. Suppose $A \subset M$ is *not* compact. Show, directly from the definition or using a theorem proved in class, that A is *not* closed.

Solution: By a theorem in class every closed subset of a compact metric space is compact, hence if A is not compact it is not closed.

Problem 4

Recall that a set S in a metric space M is connected if any separated decomposition of it, $S = A \cup B$ where $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, is 'trivial' in the sense that either A or B is empty. Show that the whole metric space M is connected if and only if the only subsets $A \subset M$ of it which are *both open and closed* are the 'trivial' cases $A = \emptyset$ and $A = M$.

Solution: Suppose first that M is connected. Let A be a subset of M which is both open and closed. Then $B = M \setminus A$ is also both open and closed and $M = A \cup B$. Since A and B are separated ($A \cap B = \emptyset$ and $\bar{A} = A$, $\bar{B} = B$) it follows, from the assumption that M is connected, that one of them is empty, so $A = \emptyset$ or $A = M$ are the only sets which are both open and closed.

Conversely, suppose that the only subsets of M which are both open and closed are M and $\emptyset$. Then let A and B be separated sets in M such that $M = A \cup B$. This means that $B = M \setminus A$ is the complement of A in M. The conditions that A and B be separated imply that $\bar{A} \cap B = \emptyset$, so $\bar{A} \subset M \setminus B = A$ hence A must be closed. Similarly B must be closed and hence A must be both open and closed. Thus one of A or B must be empty and hence, by definition, M is connected.