18.100 Midterm 2 Solutions

(1) (10 points)

a) Write down the definition of uniform continuity.

Solution. Let X and Y be metric spaces. A function $f : X \to Y$ if *uniformly continuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \quad \text{implies} \quad d_Y(f(x), f(y)) < \epsilon$$

for all $x, y \in X$.

b) Give an example of a function that is continuous but not uniformly continuous.

Solution. For instance, $f(x) = 1/x$ defined on $(0, 1)$ furnishes a continuous function that fails to be uniformly continuous.
[This can be seen as follows. Choose $\epsilon = 1$ and assume that $|f(x) - f(y)| < \epsilon = 1$ whenever $|x - y| < \delta$ for all $x, y \in (0, 1)$ and some $\delta > 0$. For $n \in \mathbb{N}$, we put $x = 1/n$ and $y = 1/2n$ so that $|x - y| = 1/2n < \delta$ for n sufficiently large. But we also have $|f(x) - f(y)| = n \geq 1$, which leads to a contradiction.]

(2) (10 points)

Let f be a continuous, differentiable function $f : \mathbb{R} \to \mathbb{R}$. If there is a real number M such that $|f'(x)| < M$ for every $x \in \mathbb{R}$, show that f is uniformly continuous.

Solution. By the mean-value theorem, we have

$$\frac{f(x) - f(y)}{x - y} = f'(\xi)$$

for all $x \neq y$ and some ξ between x and y. Thus, we infer

$$|f(x) - f(y)| \leq |f'(\xi)||x - y| < M|x - y|,$$

which, of course, is also true if $x = y$. Given $\epsilon > 0$, we choose $\delta = \epsilon/M$ to find that

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < M|x - y| = M\frac{\epsilon}{M} = \epsilon.$$

This shows uniform continuity of $f : \mathbb{R} \to \mathbb{R}$.

(3) (10 points)
Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(v + w) = f(v) + f(w)$$

for any two real numbers v and w. Assume that f is continuous at $x = 0$, show that $f(z) = f(1)\, z$ for every $z \in \mathbb{R}$.
Hint: For any $x_0 \in \mathbb{R}$, show that f is continuous at x_0 by using $f(x - x_0) = f(x) - f(x_0)$.

Solution. Clearly, $f(0) = f(0+0) = f(0) + f(0) = 2f(0)$ so that $f(0) = 0$. Therefore the claim $f(n) = nf(1)$ holds in particular if $n = 0$. Thus, by induction, we find that

$$f(n) = f((n-1)+1) = f(n-1) + f(1) = (n-1)f(1) + f(1) = nf(1) \tag{1}$$

holds for all $n \in \mathbb{N}$. Also, $f(0) = f(n-n) = f(n) + f(-n)$ shows that $f(-n) = -f(n) = -nf(1)$, so that everything extends to $\mathbb{Z}$. Moreover, we obtain $f(nt) = nf(t)$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Hence, for any $0 \neq n \in \mathbb{Z}$, we find

$$f(1) = f\big(\frac{n}{n}\big) = nf\big(\frac{1}{n}\big) \Rightarrow f\big(\frac{1}{n}\big) = \frac{1}{n}f(1). \tag{2}$$

Combining (1) and (2), we conclude that for any rational number $r = p/q$ where $p, q \in \mathbb{Z}$, $q \neq 0$,

$$f(r) = f\big(\frac{p}{q}\big) = pf\big(\frac{1}{q}\big) = \frac{p}{q}f(1) = rf(1).$$

Let x_0 be any real number. To show that f is continuous at x_0, we notice

$$f(x) - f(x_0) = f(x - x_0) \to 0 \quad \text{as } x \to x_0,$$

since f is continuous at 0. Thus $f(x) \to f(x_0)$ whenever $x \to x_0$, and hence f is continuous on $\mathbb{R}$. By continuity of f on $\mathbb{R}$ and the fact that $f(x) = xf(1)$ holds on the dense subset $\mathbb{Q} \subset \mathbb{R}$, we conclude that $f(x) = xf(1)$ holds for every $x \in \mathbb{R}$.

(4) (10 points)
Assume that f is a differentiable function on $(0, 1]$ with $|f'(x)| < 1$ for every $x \in (0, 1]$. For every natural number $n \geq 1$, define

$$a_n = f\left(\frac{1}{n}\right)$$

and show that $\lim_{n\to\infty} a_n$ exists. (Note that $f(0)$ is not defined.)
Hint: Show that (a_n) is Cauchy.

Solution. The mean-value theorem and the fact that $|f'(x)| < 1$ give us

$$|a_n - a_m| = \left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{m}\right) \right| < \left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{n} + \frac{1}{m}.$$

Thus (a_n) is Cauchy, since for every $\epsilon > 0$ the choice of an integer $N \geq 2/\epsilon$ leads to

$$|a_n - a_m| < \frac{1}{n} + \frac{1}{m} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{whenever } m, n \geq N.$$

(5) (10 points)

If f is a non-negative decreasing function defined on $[0,\infty)$, α is a strictly increasing function on $[0,\infty)$ and $f\in\mathcal{R}(\alpha)$ on any interval $[0,c]$ with $c>0$, prove that for any real numbers x and b satisfying $0<x\le b$ we have

$$f(x)\le\frac{1}{\alpha(x)-\alpha(0)}\int_0^b f\,d\alpha$$

Solution.

$$\int_0^b f\,d\alpha\ge\int_0^x f\,d\alpha\ge\int_0^x f(x)\,\,d\alpha=f(x)\left(\alpha(x)-\alpha(0)\right)$$

where we used that f is non-negative in the first inequality and that f is decreasing in the second.