

SOLUTIONS TO PS2

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Problem 1.

Proof. It is true that for any two sets A, B , the intersection $A \cap B$ is a subset of A . Now consider $\phi = A \cap A^c$. So ϕ is a subset of A for any set A . \square

Problem 2.

Proof. Notice that

$$||x| - |y|| \leq |x - y| \Leftrightarrow |x| - |y| \leq |x - y| \text{ and } |y| - |x| \leq |x - y|.$$

So we only need to prove that

$$|x| \leq |x - y| + |y| \text{ and } |y| \leq |x - y| + |x|.$$

But both of them is a consequence from the triangle inequality $|a - b| \leq |a - c| + |b - c|$. \square

Problem 3.

(a) $M = \left\{ \frac{|x|}{1+|x|} : x \in \mathbb{R} \right\}$.

Proof. Notice that

$$\frac{|x|}{1+|x|} = \frac{1}{\frac{1}{|x|} + 1}$$

so if $|x| < |y|$ then

$$\frac{|x|}{1+|x|} < \frac{|y|}{1+|y|}.$$

Thus the supremum is $\frac{1}{0+1} = 1$ and the infimum is $\frac{0}{1+0} = 0$. \square

(b) $M = \left\{ \frac{x}{1+x} \mid x > -1 \right\}$.

Proof. We can change the variable x to y ,

$$\frac{x}{1+x} = \frac{y-1}{y} = 1 - \frac{1}{y},$$

where $y = x + 1$. From $x > -1$, we have $y > 0$. Notice that

$$y \text{ increases} \Rightarrow \frac{1}{y} \text{ decreases} \Rightarrow \left(1 - \frac{1}{y}\right) \text{ increases,}$$

so the supremum is $1 - 0 = 1$ and the infimum is $-\infty$ (because for every $N > 1$ we have

$$\frac{\left(\frac{N}{1-N}\right)}{1 + \left(\frac{N}{1-N}\right)} = -N$$

and so the infimum is less than $-N$). \square

(c) $M = \{x + \frac{1}{x} | 1/2 < x < 2\}$.

Proof. It is always true that

$$\frac{a+b}{2} \geq \sqrt{ab},$$

for instance, if square both sides and rearrange, this is the same as saying $a^2 + b^2 \geq 0$. Thus, we see that

$$x + \frac{1}{x} \geq 2\sqrt{x \frac{1}{x}} = 2$$

Since setting $x = 1$ in $x + \frac{1}{x}$ we get 2, we know that $\inf M = 2$.

Suppose we have $x_1 > x_2$, consider

$$\begin{aligned} & x_1 + \frac{1}{x_1} - \left(x_2 + \frac{1}{x_2}\right) \\ &= (x_1 - x_2) + \frac{x_2 - x_1}{x_1 x_2} \\ &= \frac{(x_1 - x_2)(x_1 x_2 - 1)}{x_1 x_2} \end{aligned}$$

So if $x_1, x_2 > 1$, then

$$x_1 + \frac{1}{x_1} - \left(x_2 + \frac{1}{x_2}\right) = \frac{(x_1 - x_2)(x_1 x_2 - 1)}{x_1 x_2} > 0,$$

i.e. $x + \frac{1}{x}$ is an increasing function; if $x_1, x_2 < 1$, then

$$x_1 + \frac{1}{x_1} - \left(x_2 + \frac{1}{x_2}\right) = \frac{(x_1 - x_2)(x_1 x_2 - 1)}{x_1 x_2} < 0,$$

i.e. $x + \frac{1}{x}$ is a decreasing function. Then the sup must be obtained at the boundary of $(1/2, 2)$.

Since

$$\lim_{x \rightarrow 2} \left(x + \frac{1}{x}\right) = \lim_{x \rightarrow 1/2} \left(x + \frac{1}{x}\right) = \frac{5}{2},$$

we have $\sup M = \frac{5}{2}$. □

Problem 4.

Proof. The answer is:

n	30	42	66	78	102	114	138	174	186	70	110
p1	2	2	2	2	2	2	2	2	2	2	2
p2	3	3	3	3	3	3	3	3	3	5	5
p3	5	7	11	13	17	19	23	29	31	7	11

n	130	170	190	154	182	105	165	195
p1	2	2	2	2	2	3	3	3
p2	5	5	5	7	7	5	5	5
p3	13	17	19	11	13	7	11	13

□

Problem 5.

Proof. From $X \sim \mathbb{R}$, then there is a 1-1 mapping $\alpha : X \rightarrow \mathbb{R}$. Similarly we have a 1-1 mapping $\beta : Y \rightarrow \mathbb{N}$. So to prove $Z = X \cup Y \sim \mathbb{R}$, we only need to prove that there is a 1-1 mapping $\gamma : Z \rightarrow \mathbb{R}$. It is equivalent to show that there is 1-1 mapping $\delta : \mathbb{N} \cup \mathbb{R} \rightarrow \mathbb{R}$. The δ can be constructed by the following method:

$$\begin{aligned}\delta(x) &= x && , \text{ if } x \in \mathbb{R} \setminus \mathbb{Z}; \\ \delta(x) &= x && , \text{ if } x \in \mathbb{Z} \subset \mathbb{R} \text{ and } x \leq 0; \\ \delta(x) &= 2x && , \text{ if } x \in \mathbb{Z} \subset \mathbb{R} \text{ and } x > 0; \\ \delta(x) &= 2x + 1 && , \text{ if } x \in \mathbb{N}.\end{aligned}$$

It is easy to check that it is an 1-1 mapping. □

Problem 6.

Proof. Consider the sets

$$\begin{aligned}A_0 &= \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}, \\ A_1 &= \left\{ \frac{1}{n} + 1 \mid n \in \mathbb{N} \right\}, \\ A_2 &= \left\{ \frac{1}{n} + 2 \mid n \in \mathbb{N} \right\}.\end{aligned}$$

Then A_i has only one limit point i , for $i = 0, 1, 2$. If let $A = \bigcup_{i=0}^2 A_i$, we get a bounded set A with three limit points.

Consider the set

$$\mathcal{A} = \left\{ \frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N} \right\},$$

we check that the limit points of \mathcal{A} are precisely the points in $A_0 \cup \{0\}$. Indeed, if we fix $n_0 \in \mathbb{N}$ then the set

$$\left\{ \frac{1}{n_0} + \frac{1}{m} : m \in \mathbb{N} \right\},$$

has $\frac{1}{n_0}$ as a limit point and is a subset of \mathcal{A} , hence \mathcal{A} has $\frac{1}{n_0}$ as a limit point, for any n_0 in \mathbb{N} . Also $A_0 \subseteq \mathcal{A}$ so zero is a limit point of \mathcal{A} . To see that there are no other limit points, pick a point $x \in \mathbb{R}$ that is not equal to $\frac{1}{n}$ for any $n \in \mathbb{N}$, we show that x is not a limit point of \mathcal{A} . We can find $N \in \mathbb{N}$ such that

$$\frac{1}{N} < x < \frac{1}{N-1}$$

Pick $\varepsilon > 0$ small enough so that

$$\frac{1}{N} < x - \varepsilon < x < x + \varepsilon < \frac{1}{N-1}$$

and notice that there are at most finitely many elements of \mathcal{A} in $(x - \varepsilon, x + \varepsilon)$. Here is one way to see this: if n and m are both bigger than $2N$ then $\frac{1}{n} + \frac{1}{m} < \frac{1}{N}$, if $n < N$ then $\frac{1}{n} + \frac{1}{m} > \frac{1}{N-1}$, while if $2N \geq n > N$ then

$$\frac{1}{N} < \frac{1}{n} + \frac{1}{m} \iff -\frac{1}{m} < \frac{1}{n} - \frac{1}{N} = \frac{N-n}{nN} \iff m < \frac{nN}{n-N},$$

finally if $n = N$, and m is large enough then $\frac{1}{n} + \frac{1}{m} < x - \varepsilon$. So there are finitely many possible pairs (n, m) with $x - \varepsilon < \frac{1}{n} + \frac{1}{m} < x + \varepsilon$.

Since there are only finitely many elements of \mathcal{A} inside $(x - \varepsilon, x + \varepsilon)$ we can find $k \in \mathbb{N}$ so that $(x - \frac{\varepsilon}{k}, x + \frac{\varepsilon}{k})$ contains no element of \mathcal{A} except possibly x itself. This proves that x is not a limit point of \mathcal{A} . □

Problem 7.

Proof. (a) The points in E^0 are interior points of E , to show that E^0 is open we need to show that they are interior points of E^0 . Given $x \in E^0$, by definition, there exist an open ball $x \in B_r(x) \subset E$. Consider an open ball $B_{r/3}(x) \subset B_r(x)$. Then for any point $y \in B_{r/3}(x)$, $B_{r/3}(y) \subset B_r(x) \subset E$, so $y \in E^0$. Then $B_{r/3}(x)$ is an open ball in E^0 . So x is an interior point of E^0 .

(b) If $E = E^0$, from (a) we know that E is open. Conversely, if E is open, all points in E are interior points, so $E \subset E^0$. From $E^0 \subset E$ we have $E = E^0$.

(c) Since G is open, so for any point $g \in G$, we have an open ball $B_r(g) \subset G \subset E$. So g is also an interior point of E . Then $G \subset E^0$. □