Fourier Series and Their Applications

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Abstract

Fourier series are of great importance in both theoretical and applied mathematics. For orthonormal families of complex-valued functions $\{\phi_n\}$, Fourier Series are sums of the ϕ_n that can approximate periodic, complex-valued functions with arbitrary precision. This paper will focus on the Fourier Series of the complex exponentials. Of the many possible methods of estimating complex-valued functions, Fourier series are especially attractive because uniform convergence of the Fourier series (as more terms are added) is guaranteed for continuous, bounded functions. Furthermore, the Fourier coefficients are designed to minimize the square of the error from the actual function. Finally, complex exponentials are relatively simple to deal with and ubiquitous in physical phenomena. This paper first defines generalized Fourier series, with an emphasis on the series with complex exponentials. Then, important properties of Fourier series are described and proved, and their relevance is explained. A complete example is then given, and the paper concludes by briefly mentioning some of the applications of Fourier series and the generalization of Fourier series, Fourier transforms.

1 Introduction and Background Information

In the mid-eighteenth century, physical problems such as the conduction patterns of heat and the study of vibrations and oscillations led to the study of Fourier series. Of central interest was the problem of how arbitrary real-valued functions could be represented by sums of simpler functions. As we shall see later, a *Fourier series* is an infinite sum of trigonometric functions that can be used to model real-valued, periodic functions.

We shall begin by giving a brief description of the trigonometric polynomials, and especially of their relation to the complex exponentials. Let us define:

$$cos(x) = \frac{1}{2}[e^{ix} + e^{-ix}], \tag{1}$$

$$sin(x) = \frac{1}{2i}[e^{ix} - e^{-ix}]. \tag{2}$$

Another way to express this is that $cos(x)$ is the real part of e^{ix} and that $sin(x)$ is the imaginary part. Let us quickly show that both functions are periodic with period 2π. To do so, we just need to show that $e^{i(x+2\pi)} = e^{ix}$, and the periodicity of $sin(x)$ and $cos(x)$ follow by definition.

$$\begin{aligned} e^{i(x+2\pi)} &= e^{ix}e^{2i\pi} \\ &= e^{ix}e^{(i\pi)^2} \\ &= e^{ix}(-1)^2 \\ &= e^{ix} \end{aligned}$$

We now move to the definition of a *trigonometric polynomial.* For complex numbers $\{a_0, a_1, a_2 \ldots\}$ and $\{b_1, b_2, b_3 ...\}$, and real x, we define a *trigonometric polynomial* to be a finite sum of the form:

$$f(x) = a_0 + \sum_{n=0}^{N}(a_n cos(nx) + b_n sin(nx)). \tag{3}$$

Now, for $n = 1, 2, 3 \ldots$, define $c_n = \frac{1}{2}(a_n - ib_n)$, $c_{-n} = \frac{1}{2}(a_n + ib_n)$. Also, define $c_0 = a_0$. Then from the above identity, we can also write a trigonometric polynomial in equivalent form by:

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx}. \tag{4}$$

Now consider the function $\frac{e^{inx}}{in}$. Clearly, both $f_1(x) = e^{inx}$ and $f_2(x) = \frac{e^{inx}}{in}$ both have periods of 2π. Furthermore, we know that $\int_{-\pi}^{\pi} e^{(inx)}dx = \frac{e^{inx}}{in} = f_2(x)$. Then, since $f_2(x) = f_2(x + 2\pi)$, $\int_p^{p+2\pi} f_1(x) = 0$ if $n = 1, 2, 3 \ldots$, and if $n = 0$, $\int_p^{p+2\pi} f_1(x) = 2\pi$.

Now let us multiply equation (4) above by e^{-imx}, where m is any integer. We obtain the following:

$$\begin{aligned} e^{-imx}f(x) &= e^{-imx}\sum_{n=-N}^{N} c_n e^{inx} \\ &= \sum_{n=-N}^{N}[(c_n e^{inx})(e^{-imx})] \\ &= \sum_{n=-N}^{N}(c_n e^{i(n-m)x}) \end{aligned}$$

Consider two cases. First, suppose that $|m| > |N|$. In this case, for all $-|N| \leq n \leq |N|$, $n - m \neq 0$. Thus, in this case, $\int_{-\pi}^{\pi} e^{i(n-m)x}dx = 0$.

Now, suppose that $|m| \leq |N|$. In this case, there is exactly one n such that $n \in Z$, $-N \leq n \leq N$, and that is $n = m$. Since $\int_{-\pi}^{\pi} e^{inx} dx = 2\pi$ if and only if $n = 0$,

$$\int_{-\pi}^{\pi} e^{-imx} f(x) dx = \int_{-\pi}^{\pi} c_n e^{i(m-n)x} dx,$$

where $m = n$. Thus,

$$\int_{-\pi}^{\pi} e^{-imx} f(x) dx = 2\pi c_n, \text{ and}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx} f(x) dx.$$

Since $m = n$,

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx} f(x) dx \tag{5}$$

We now define a *trigonometric series* to be of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{6}$$

where the Nth partial sum is

$$\sum_{n=-N}^{N} c_n e^{inx}. \tag{7}$$

Furthermore, for a given function $f(x)$, we shall define the *Fourier series of f(x)* as the trigonometric series with coefficients of the form given in equation (5).

2 General Fourier Series

Before focusing on Fourier series with trigonometric functions, we shall give a description of *general Fourier functions.*

We start with the notion of *orthogonal systems of functions.* Let $\{\phi_1, \phi_2, \phi_3 ...\}$ be a series of complex functions. We say that $\{\phi_n\}$ is an *orthogonal system of functions on* [*a*,*b*] if, for all integers $m \neq n$,

$$\int_a^b \phi_m(x) \bar{\phi}_n(x) dx = 0, \tag{8}$$

As a further note, if for all integers $m > 0$,

$$\int_a^b \phi_m(x) \bar{\phi}_m(x) dx = 1,$$

we say that $\{\phi_n\}$ is an *orthonormal system of functions.*

We have already seen that the functions e^{inx}, $n = 1, 2, 3...$ form an orthogonal system of functions on $[-\pi, \pi]$, since $\bar{e}^{inx} = e^{-inx}$, and for $m \neq n$, $\int_{-\pi}^{\pi} e^{inx}e^{-imx} = 0$. We now define the *Fourier coefficients with respect to* $\{\phi_n(x)\}$ as follows:

$$c_n = \int_a^b f(x)\bar{\phi}_n(x). \tag{9}$$

, where $\bar{\phi}_n(x)$ is the complex conjugate of the complex-valued function $\phi(x)$.

In terms of generalized Fourier series, define the *Fourier series of f with respect to* $\{\phi_n(x)\}$ to be

$$\sum_{n=1}^{\infty} c_n\phi_n(x) \tag{10}$$

To see how our definition for the Fourier series with respect to trigonometric functions matches this pattern, let

$$\begin{aligned} \{\phi_n(x)\} &= \{\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x)\} \\ &= \{e^{ix}, e^{(-ix)}, e^{2ix}, e^{(-2ix)}\} \end{aligned}$$

and let $\phi(x) = e^{inx}$, $\bar{\phi}(x) = e^{(-inx)}$.

3 Some Properties of Fourier series

We now present a few important properties of Fourier series from Walter Rudin's Principles of Mathematical Analysis.

Theorem 8.11 in Rudin: Suppose that $\{\phi_n\}$ is an orthonormal system of functions on the interval $[-\pi, \pi]$. Suppose that we have two sets of complex numbers, c_n and d_n, $n = 0, 1, 2, 3...$ and c_n are the Fourier Coefficients for $\{\phi_n\}$, as defined in equation (9). Now, consider two series of functions,

$$s_N(f, x) = \sum_{n=1}^{N} c_n\{\phi_n\}, \tag{11}$$

which is the N^{th} partial sum of the Fourier series for f, and

$$t_N(f, x) = \sum_{n=1}^{N} d_n\{\phi_n\}. \tag{12}$$

Then,

$$\int_a^b |f - s_n(f, x)|^2 dx \leq \int_a^b |f - t_n(f, x)|^2 dx. \tag{13}$$

This theorem indicates that, for some periodic function f and some orthonormal system of functions $\{\phi_n\}$, the Fourier series provides the least total squared-error approximation.

Proof: Let $\{\phi_n(x)\}$ be orthonormal on the interval $[a, b]$. Consider:

$$\textstyle\int_a^b f\bar{t}_n = \int_a^b f \sum_1^n \bar{d}_n \bar{\phi}_n$$

by the definition of t_n. We can also write the above as:

$$\textstyle\int_a^b f \sum_1^n \bar{d}_n \bar{\phi}_n = \sum_1^n \int_a^b f \bar{d}_n \bar{\phi}_n$$

Since $\int_a^b f\bar{\phi}_n = c_n$, we can once again rewrite the above expression as:

$$\int_a^b f\bar{t}_n = \sum_1^n c_n \bar{d}_n \tag{14}$$

Now, consider the integral from a to b of $|t_n|^2$. Since

$$|t_n|^2 = t_n \bar{t}_n$$

and

$$\textstyle t_n = \sum_1^n d_m \phi_m$$

$$\textstyle\bar{t}_n = \sum_1^n \bar{d}_m \bar{\phi}_m,$$

we have:

$$\textstyle\int_a^b |t_n|^2 = \int_a^b \sum_1^n d_m \phi_m \sum_1^n \bar{d}_k \bar{\phi}_k,$$

which we may rewrite as:

$$\int_a^b |t_n|^2 = \sum_1^n d_m \phi_m \int_a^b \sum_1^n \bar{d}_k \bar{\phi}_k \tag{15}$$

Since $\{\phi_n\}$ is an orthonormal system of functions on $[a, b]$, according to equation (8), we can rewrite the above as:

$$\textstyle\int_a^b |t_n|^2 = \sum_1^n d_m \bar{d}_m,$$

which we can again rewrite as:

$$\int_a^b |t_n|^2 = \sum_1^n |d_m|^2 \tag{16}$$

Now, consider the total squared error between f and t_n, $\int_a^b |f - t_n|^2$. We first rewrite it as:

$$\int_a^b |f-t_n|^2 = \int_a^b (f-t_n)(\overline{f-t_n})$$

Furthermore, we know that

$$\overline{(f-t_n)} = \bar{f} - \bar{t}_n,$$

so that

$$\begin{aligned}\int_a^b |f-t_n|^2 &= \int_a^b (f-t_n)(\bar{f}-\bar{t}_n)\\ &= \int_a^b (f\bar{f} - f\bar{t}_n - \bar{f}t_n + t_n\bar{t}_n)\\ &= \int_a^b (f^2 - f\bar{t}_n - \bar{f}t_n + t_n^2)\end{aligned}$$

By equations (14) and (16) above, we can write:

$$\int_a^b |f-t_n|^2 = \int_a^b (f^2) - \sum_1^n (c_m\bar{d}_m) - \sum_1^n (\bar{c}_m d_m) + \sum_1^n (d_m\bar{d}_m) \tag{17}$$

Since

$$\begin{aligned}|d_m - c_m|^2 &= (d_m - c_m)(\bar{d}_m - \bar{c}_m)\\ &= |d_m|^2 + |c_m|^2 - (c_m\bar{d}_m) - (\bar{c}_m d_m)\end{aligned}$$

We can rewrite the above equation as:

$$\int_a^b |f-t_n|^2 = \int_a^b (f^2) - \sum_1^n |c_m|^2 + \sum_1^n |d_m - c_m|^2 \tag{18}$$

From this above equation, we can see that the total error squared is minimized when $d_m = c_m$, for $m = 1, 2, 3 \ldots$.

Theorem 8.12 in Rudin: Assume all the notation used in the description of Theorem 8.11. Consider the sequence of terms $\{c_n\} = c_1, c_2, c_3 \ldots$. The series $\sum_1^n (c_m)$ converges absolutely (in other words, the series $\sum_1^n |c_m|$ converges).

Proof: In equation (16) above, substitute c_n for d_n. We obtain the following:

$$\int_a^b |s_n|^2(x)dx = \sum_1^n |c_m|^2 \tag{19}$$

The above step will not be necessary, but it is interesting to point out that the integral of the absolute value of any n^th Fourier trigonometric polynomial will be less than the integral of the absolute value of the function f.

Now consider equation (18). Since we know that $\int_a^b |f - t_n|^2 \geq 0$, it follows that

$$\sum_1^n |c_m|^2 \leq \int_a^b |f(x)|^2 dx \tag{20}$$

If we let n go to infinity, we see that:

$$\sum_1^\infty |c_n|^2 \leq \int_a^b |f(x)|^2 \tag{21}$$

From our study of convergent series, this also implies that

$$lim_{n \to \infty} c_n = 0$$

This result is fairly important because it shows us that it is the first terms of a Fourier series that are most important, and that the Fourier coefficients become arbitrarily small. In terms of simulations, this implies that a few terms may provide a very good model of a function.

Theorem 8.14 in Rudin: For a periodic function $f(x)$, suppose that for some x, there is a $\delta > 0$ and some finite, real M such that if $-\delta < t < \delta$, then $|f(x+t) - f(x)| \leq M|t|$. Then, the value of the infinite Fourier series $s_n(f, x)$ evaluated at x converges to $f(x)$ (evaluated at x) as n approaches infinity.

This theorem talks about the "pointwise convergence" of a Fourier series. At all points a with the property above, the series of Fourier polynomials converges pointwise to $f(a)$ at a. An interesting consequence of this result is that for some function $f(x)$ that is uniformly continuous on some segment (a, b), the Fourier series will converge to the function $f(x)$ in value for all x in that segment. A stronger result that describes the *uniform convergence* of the Fourier series follows.

Theorem 8.15 in Rudin: This is another theorem of convergence, although it does not mention Fourier series explicitly. Suppose that $f(x)$ is continuous with period 2π. Then, for any $\epsilon > 0$, there is some trigonometric polynomial such that $|P(x) - f(x)| < \epsilon$ for all x.

Another way of stating this is that given any continuous, periodic, complex-valued function f, there is some sequence of trigonometric polynomials that converges uniformly to f. This is basically a direct application of the Stone-Weierstrauss Theorem for complex-valued functions. The metric space K of the complex exponentials is the unit circle, which is closed and bounded in R^2, and therefore compact. The family of trigonometric functions is an algebra of functions, since it is closed under addition, multiplication, and scalar multiplication. It clearly separates all points, and vanishes at no points. Furthermore, it is self adjoint — for every trigonometric polynomial $f(x)$, there is a trigonometric polynomial $\bar{f}(x)$. Thus, the family of trigonometric polynomials is dense in K.

4 An Example

We now apply the Fourier series to a few basic examples. Let us first consider the Fourier series for $f_1(x)$, which is a continuous function with period 2π, and whose value on the interval $[-\pi, \pi]$ is x.

We know that the Fourier coefficients c_n are given by:

$$c_n = \tfrac{1}{2\pi} \textstyle\int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Since

$$\textstyle\int_{-\pi}^{\pi} x e^{-inx} dx = (\tfrac{1}{-in} x e^{-inx} + \tfrac{1}{n^2} e^{-inx})|_{-\pi}^{\pi}$$

we know that

$$\begin{aligned} c_n &= \frac{1}{2\pi} \frac{e^{-inx}}{n^2} (inx + 1)|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \frac{e^{-in\pi}}{n^2} (in\pi + 1) - \frac{e^{in\pi}}{n^2} (-in\pi + 1) \\ &= \frac{1}{2\pi} \left(\frac{(cos(n\pi) - isin(n\pi))(in\pi + 1) - (cos(n\pi) + isin(n\pi))(-in\pi + 1)}{n^2} \right) \\ &= \frac{1}{2\pi} \left(\frac{in\pi cos(n\pi) - isin(n\pi) + in\pi cos(n\pi) - isin(n\pi)}{n^2} \right) \\ &= \frac{1}{2\pi} \frac{2(ncos(n\pi)\pi - sin(n\pi))i}{n^2} \end{aligned}$$

and

$$\begin{aligned} c_{-n} &= \frac{1}{2\pi} \frac{e^{inx}}{n^2} (-inx + 1)|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \frac{e^{in\pi}}{n^2} (-in\pi + 1) - \frac{e^{-in\pi}}{n^2} (in\pi + 1) \\ &= \frac{1}{2\pi} \frac{2(-ncos(n\pi)\pi + sin(n\pi))i}{n^2} \end{aligned}$$

This almost gives us all the information we need to determine the Fourier series. But we first need to calculate c_0, since the method of integrating e^0 is different. So,

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^0 \\ &= \frac{1}{2\pi} \frac{1}{2} (\pi^2 - \pi^2) \\ &= 0 \end{aligned}$$

Now we can construct the Fourier series for $f_1(x)$. For example, let's take the first 9 terms (or $n = 0, 1, 2, 3, 4$).

We have: $c_0 = 0$, $c_1 = -i$, $c_{-1} = i$, $c_2 = \frac{i}{2}$, $c_{-2} = -\frac{i}{2}$, $c_3 = -\frac{i}{3}$, $c_{-3} = \frac{i}{3}$, $c_4 = \frac{i}{4}$, $c_{-4} = -\frac{i}{4}$, and the $4th$ partial sum of the Fourier series of $f_1(x)$ is given by:

$$s_N(f_1, x) = 0 - ie^{ix} + ie^{-ix} + (\frac{i}{2}e^{2ix}) - \frac{i}{2}e^{-2ix} - \frac{i}{3}e^{3ix} + \ldots$$

We can now combine terms raised to the negative exponents of each other, and rewrite each complex exponential in terms of its sine and cosine terms. Doing so, we obtain the formula

$$s_N(f_1, x) = 0 + 2sin(x) - sin(2x) + \tfrac{2}{3}sin(3x) - \tfrac{1}{2}sin(4x)$$

Graphing this function, we can indeed see the shape of $f_1(x)$ beginning to appear.

Figure 1: Fourier Series with First 5 Terms

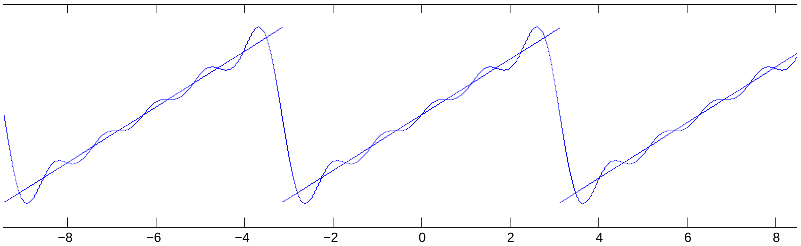


Figure 2: Fourier Series with First 15 Terms

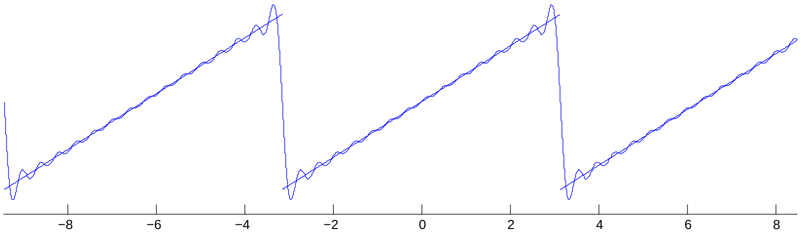
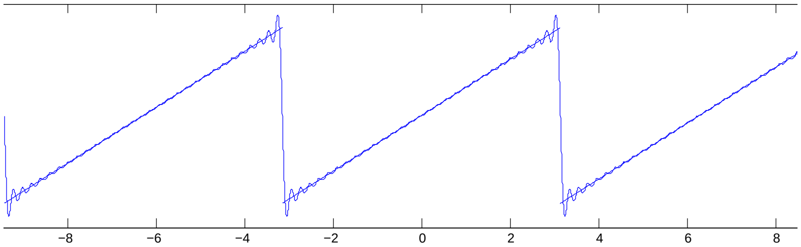


Figure 3: Fourier Series with First 30 Terms



Furthermore, if we calculate the total squared error on the interval $[-\pi, \pi]$, we find that:

$$\int_{-\pi}^{\pi}(x - s_N(f_1, x))^2 \approx 2.781.$$

Indeed, if we use the first 41 terms of the Fourier series (the $20th$ partial sum), the function appears much more similar to $f_1(x)$, and the square error from $-\pi$ to π decreases dramatically.

Sometimes, leaving the coefficients in the forms $\{c_0, c_1, c_2...\}$ is not very convenient. We will now use some properties of complex exponentials to create a nicer form for the Fourier series.

First, since the term associated with c_0 in the Fourier series is c_0e^0, we can replace it with some real constant. Let's call this constant $a_0 = c_0$. For c_n and c_{-n}, $n = 1, 2, 3...$, recall the definitions that

$$c_n = \frac{1}{2}(a_n - ib_n)$$
$$c_{-n} = \frac{1}{2}(a_n + ib_n)$$

Then,

$$a_n = c_n + c_{-n}$$
$$b_n = i(c_n - c_{-n})$$

Since we know that c_{-n} is the complex conjugate of c_n, we can say that a_n is twice the real part of c_n (or c_{-n}), and b_n is negative two times the real coefficient of the imaginary part of c_n (or c_{-n}).

These relations let us put the Fourier series into an alternate form, given by:

$$f(x) = a_0 + \sum_{n=0}^{N}(a_n cos(nx) + b_n sin(nx)) \tag{22}$$

where a_n and b_n are given as above.

5 Applications of Fourier series

To recapitulate, Fourier series simplify the analysis of periodic, real-valued functions. Specifically, it can break up a periodic function into an infinite series of sine and cosine waves. This property makes Fourier series very useful in many applications. We now give a few.

Consider the very common differential equation given by:

$$x''(t) + ax'(t) + b = f(t) \tag{23}$$

This equation describes the motion of a damped harmonic oscillator that is driven by some function $f(t)$. It can be used to model an extensive variety of physical phenomena, such as a driven mass on a spring, an analog circuit with a capacitor, resistor, and inductor, or a string vibrated at some frequency. There are two parts to the solution of equation (25). The first part is a *transient* that fades away (generally) fairly quickly. When the transient is gone, what remains is the *steady-state solution*. This is what we will concern ourselves with.

If $f(t)$ is a sinusoid, then the solution is also a sinusoid which is not very difficult to find. The problem is that the driver is generally not a simple sinusoid, but some other periodic function. In electronics, for example, a common driving voltage function is the square wave $s(t)$, a periodic function (whose period we shall say is 2π) such that $s(t) = 0$ for $-\pi \leq t < 0$ and $s(t) = 1$ for $0 \leq t < \pi$.

The physical property of oscillating systems that makes Fourier Analysis useful is the property of *superposition* - in other words, suppose the driving force

$f_1(t)$, along with some initial conditions, produces some steady state solution $x_1(t)$, and that another driving force, $f_2(t)$ produces the steady state solution $x_2(t)$. Then the driving force $f_3(t) = f_1(t) + f_2(t)$ produces the steady-state response $x_3(t) = x_1(t) + x_2(t)$.

Then, since we can represent any period driving function as a Fourier series, and it is a simple matter to find the steady state solution to a sinusoidally-driven oscillator, we can find the response to the arbitrary driving function $f(x) = a_0 + \sum(a_n cos(nx) + b_n sin(nx))$.

So suppose we had our square wave equation, where $f(t)$ is the square wave function. We could then decompose the square wave into sinusoidal components as follows:

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} s(x)e^{-inx} \\ &= \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx} \\ &= \frac{i}{2n\pi}(e^{in\pi}-1) \\ c_{-n} &= \frac{-i}{2n\pi}(e^{-in\pi}-1) \end{aligned}$$

and then just combine the c_n and c_{-n} terms as before. The result would be an infinite sum of *sin* and *cos* terms of the form in equation (2). The steady-state response of the system to the square wave would then just be the sums of the steady-state responses to the sinusoidal components of the square wave.

The basic equations of the Fourier series led to the development of the Fourier transform, which can decompose a non-periodic function much like the Fourier series decomposes a periodic function. Because this type of analysis is very computation-intensive, different *Fast-Fourier Transform* algorithms have been devised, which lower the order of growth of the number of operations from order(N^2) to order($n\log(n)$).

With these new techniques, Fourier series and Transforms have become an integral part of the toolboxes of mathematicians and scientists. Today, it is used for applications as diverse as file compression (such as the JPEG image format), signal processing in communications and astronomy, acoustics, optics, and cryptography.

References

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