The final

This exam is closed book, no books, papers or recording devices permitted. You may use theorems from class, or the book, provided you can recall them correctly.

With short solutions and comments. I hope I did not damage too many delicate egos with this final!

Problem 1

Suppose $f \in L^2([-\pi,\pi])$ and $f(x) \geq 0$ for all $x \in [-\pi,\pi]$. Show that $\int_a^b f(x)dx = 0$ for all $a \leq b$, with $a, b \in [0,1] \cap \mathbb{Q}$ implies that $f = 0$ almost everywhere with respect to Lebesgue measure.

Generally well done.

Ans:- From a standard result, $f = 0$ a.e. in $[a, b]$ for every $a, b \in [-\pi,\pi] \cap \mathbb{Q}$. The subset of $[-\pi,\pi]$ on which $f(x) > 0$ is therefore the union of a countable collection of sets of measure zero, so also of measure zero. Hence $f = 0$ almost everywhere in $[-\pi,\pi]$.

Problem 2

Suppose A is a bounded and self-adjoint operator on a Hilbert space H, show that if $\langle Au, u\rangle = 0$ for all $u \in H$ then $A = 0$.

Only one person got this right. I did not say compact, so you cannot use the spectral theory of compact operators. No, all bounded operators are not compact etc.

Ans:- For any $u, v \in H$, expanding using sesquilinearity (and linearity of A) gives

$$0 = \langle A(u+v), u+v\rangle + i\langle A(u+iv), u+iv\rangle = 2\langle Au, v\rangle.$$

Setting $v = Au$ implies that $\|Au\| = 0$ for all $u \in H$, that is $A = 0$ as an operator.

Problem 3

Give an example of an element of $L^2(\mathbb{R})$ which is not the Fourier transform of an element of $L^1(\mathbb{R})$.

Of course, I confused people by giving a harder version of this on the practice final.

Ans. If the Fourier transform of $f \in L^2(\mathbb{R})$ is in L^1 then it must be the inverse Fourier transform of an L^1 function, hence continuous. Thus it suffices to find an L^2 function which is not continuous, say the characteristic function of $[0,1]$.

Problem 4

If $u \in L^1(\mathbb{R})$ show that

$$v(x) = \sum_{j\in\mathbb{Z}} u(x - 2\pi j), \;\; x \in [-\pi, \pi]$$

converges to an element of $L^1([-\pi,\pi])$.

Generally well done although quite a few people tried to use Fourier series.

Ans:- By the translation-invariance of Lebesgue measure and countable additivity, the terms, $v_j = u(x - 2\pi j)$, for $x \in [-\pi,\pi]$, in the series satisfy

$$\sum_{j\in\mathbb{Z}} \int_{[-\pi,\pi]} |v_j| dx = \sum_{\mathbf{R}} |u| dx < \infty.$$

Thus, the series $\sum_j v_j$ is Cauchy in $L^1([-\pi,\pi])$ and hence converges by the completeness of $L^1([-\pi,\pi])$.

Problem 5

Show that there is no function $u \in L^2([-\pi,\pi])$ satisfying

$$\int_{[-\pi,\pi]} u(x)\sin(kx)dx = k^{-\frac{1}{2}} \;\forall\; k \in \mathbb{N}.$$

Mostly people got this one.

Ans:- If there is such a function its real part also satisfies the condition, so we can assume it is real. Then the integral in ([*]) is 2π times the imaginary part of the kth Fourier coefficient, thus $|c_k|^2 > 4\pi^2 k$ which series diverges, violating Bessel's inequality.

Problem 6

Show that the function $f(x) = |x|\exp(-x^2)$ is in $L^2(\mathbb{R})$ and that its Fourier transform is infinitely differentiable. Is $\hat{f}$ a Schwartz function?

Many people thought this function was Schwartz, it is not infinitely differentiable and that is one of the requirements!

Ans: Certainly $f \in L^2(\mathbb{R})$ and we know (if you like from class) that $x^p f \in L^1(\mathbb{R})$ for every $p \in \mathbb{N}$. Since follows that $\frac{d^p}{d\xi^p}\hat{f}(\xi)$ is continuous for every p. No, it is not Schwartz.

Problem 7

Suppose $f \in L^2(\mathbb{R})$ is such that $xf(x) \in L^2(\mathbb{R})$ and the Fourier transform is such that $\xi\hat{f}(\xi) \in L^2(\mathbb{R})$. Show that there exists a function $g \in L^2(\mathbb{R})$ such that

$$\int_{\mathbf{R}} f(x)\left(\frac{d}{dx}\phi(x) + x\phi(x)\right)dx = \int_{\mathbf{R}} g(x)\phi(x)dx \ \forall\ \phi \in \mathcal{S}(\mathbb{R}).$$

Many people got this one at least more-or-less right. However, many of the arguments to justify the integration by parts were dubious to say the least.

Ans:- Since $xf(x) \in L^2(\mathbb{R})$ is given, we only need to find $g_1 \in L^2(\mathbb{R})$ satisfying

$$\int_{\mathbf{R}} f(x)\left(\frac{d}{dx}\phi(x)\right)dx = \int_{\mathbf{R}} g_1(x)\phi(x)dx \ \forall\ \phi \in \mathcal{S}(\mathbb{R})$$

and then $g = g_1 + xf(x)$ will solve the actual problem. If we set $\phi = \hat{\psi}$, as we can with $\psi \in \mathcal{S}(\mathbb{R})$ uniquely determined, then $d\phi/dx = \widehat{i\xi\psi(\xi)}$ so

$$\begin{aligned}\int_{\mathbf{R}} f(x)\left(\frac{d}{dx}\phi(x)\right)dx &= \\ i\int_{\mathbf{R}} f(x)\widehat{\xi\psi}(x)dx &= i\int_{\mathbf{R}} \hat{f}(\xi)\xi\psi(\xi)d\xi \\ &= \int_{\mathbf{R}} \hat{g}_1(\xi)\psi(\xi)d\xi = \int_{\mathbf{R}} g_1(x)\phi(x)dx\end{aligned}$$

where by definition $g_1 \in L^2(\mathbb{R})$ is the function with Fourier transform equal to $i\xi\hat{f}(\xi)$ - in L^2 by assumption.

Problem 8

If $f \in L^2([0,1])$, show that

$$(Au)(x) = \int_0^x f(t)u(t)dt, \ x \in [0,1],$$

defines a compact operator $A : L^2([0,1]) \longrightarrow L^2([0,1])$.

One can argue directly as I did in the practice exam and show that Au has equismall tail with respect to the Fourier basis when $\|u\| \leq 1$. Or

Ans:- If one sets $F(x,t) = f(t)$ when $t \in [0,1], \ x \geq t$ and zero otherwise one gets a function in $L^2([0,1]^2)$ sucht that

$$Au(x) = \int_{[0,1]} F(x,t)u(t)dt.$$

Thus, from result in class this operators is Hilbert-Schmid, hence compact.

Problem 9

Show that there exists an element $f \in \mathcal{S}(\mathbb{R})$ which has $\int_{\mathbb{R}} |f|^2 dx = 1$ but $\int_{\mathbb{R}} x^k f(x)dx = 0$ for every k.

Rather a lot of wild things said here.

Ans:- If $f \in \mathcal{S}(\mathbb{R})$ then its Fourier transform is Schwartz and conversely. We also know that

$$\frac{d^k}{d\xi^k}\hat{f}(0) = (-i)^k \int_{\mathbb{R}} x^k f(x)dx.$$

Thus we just have to arrange that $\hat{f}$ and ALL its derivatives vanish at the origin. We do know that there is a non-trivial Schwartz function which vanishes outside $[1,2]$ for instance. Taking this as the Fourier transform of cf and then choosing the positive constant so that $\|f\|_{L^2} = 1$ solves the problem.