Countable additivity of the integral under decomposition of the domain.

Corollary. *If $f \geq 0$ measurable, we can define $\mu_f(E)$ for $E \subset \mathcal{F}$ as*

$$\mu_f(E) = \int_E f d\mu$$

This is a countably additive measure, because of countable additivity under decomposition of the domain. So we can construct new measures from old.

Example. For Borel subsets on the real line $E \subset B(\mathbb{R})$ we can define a probability measure

$$\mu_G(E) = \frac{1}{\sqrt{2\pi}} \int_E \exp(-\frac{x^2}{2}) d\mu_L$$

so that $\mu_G(\mathbb{R}) = 1$. We can get measures from any positive function. An important question in analysis is how measures are related (Radon-Nikodym)

Proposition. ***(Monotone Convergence Theorem)*** *We are in the measure space $(X, \mathcal{F}, \mu)$. Suppose f_j is a sequence of measurable functions $f_j : X \to [0, \infty]$, $f_j(x)$ increasing for each x. So $\lim_{i\to\infty} f_i(x) = f(x)$ is a measurable function, then*

$$\int_E \lim_{j\to\infty} f_j d\mu = \lim_{j\to\infty} \int_E f_j d\mu$$

Proof. First,

$$f_j \leq f \Rightarrow \int_E f_j d\mu \leq \int_E f_{j+1} d\mu \leq \int_E f d\mu \Rightarrow \lim_{j\to\infty} \int_E f_j d\mu \leq \int_E f d\mu$$

(sort of follows because our integral is one from below, but we have to show that the integral somehow works the other ways as well)

Suppose $0 \leq s \leq f$ is simple measurable. Suppose $0 < c < 1$ is a constant. Define $E_n = \{x \in E | f_n(x) \geq cs(x)\}$. So then $E_{n+1} \supset E_n$, and $\bigcup_{n=1}^{\infty} E_n = E$.Then

$$\lim_{j\to\infty} \int_E f_j d\mu \geq \int_E f_n d\mu \geq \int_{E_n} f_n d\mu \geq \int_{E_n} cs d\mu$$

since n is independent of j we can take limits as $n \to \infty$, so

$$\lim_{j\to\infty} \int_E f_j d\mu \geq c \int_E s d\mu \Rightarrow \lim_{j\to\infty} \int_E f_j d\mu \geq \int_E s d\mu$$

but this is true for any s which approximates f, so in fact

$$\lim_{j\to\infty} \int_E f_j d\mu \geq \int_E f d\mu$$

□

There exists a sequence $0 \leq s_j$ of simple functions such that $s_j \uparrow f(x)$, so now we've just shown that

$$\int_E f d\mu = \lim_{n\to\infty} \int_E s_n d\mu$$

This is stronger than our "sup" definition, because you need just one sequence approaching f (increasing).

Theorem. ***Linearity*** *f and g non-negative and measurable and $c \geq 0$, then*

$$\int_E cf \ d\mu = c \int_E f d\mu \qquad \int (f+g) d\mu = \int f d\mu + \int g d\mu$$

Proof. We already know that first one. Now we choose a sequence of simple functions $0 \leq s_j \leq f(x)$ such that $s_j(x) \uparrow f(x)$, $\forall x \in X$ and similarly t_j with $t_j \uparrow g$, and we are done, because we know linearity for simple functions and we can take limits □

Corollary. *If f_j is a sequence of measurable non-negative functions, and $\sum_{i=1}^{\infty} f_i(x) = f(x)$. Then*

$$\int_E \sum_{i=1}^{\infty} f_i(x) = \sum_{i=1}^{\infty} \int_E f_i d\mu$$

Proof. Let $F_N = \sum_{i=1}^{N} f_i$, then

$$\int_E F_N d\mu = \sum_{i=1}^{N} \int_E f_i d\mu \Rightarrow \int_E f d\mu = \sum_{i=1}^{\infty} \int_E f_i d\mu$$

□

Definition. A measurable function $f : X \to [-\infty, \infty]$ is integrable, $f \in \mathcal{L}(\mu, E)$ if

$$\int_E f_+ d\mu < \infty \qquad \int_E f_- d\mu < \infty$$

that is $f \in \mathcal{L}(E, \mu)$ if and only if f is measurable and $\int_E |f| d\mu < \infty$.

If f is integrable, $f \in \mathcal{L}(E, \mu)$, then

$$\int_E |f| d\mu = \int_E f_+ d\mu - \int_E f_- d\mu$$

But now we have to go back and check that $\mathcal{L}$ is a linear space.

So we check that for $f \in \mathcal{L}(E, \mu)$, then $cf \in \mathcal{L}(E, \mu)$.

Also, we need to show that $h = f + g$, $f, g \in \mathcal{L}(E, \mu)$ then

$$\int_E h \; d\mu = \int_E f \; d\mu + \int_E g \; d\mu$$

do this by checking where f, g and h have a fixed sign, and then sum up those regions.

Now we discuss more theorems on integrals.

Theorem. ***Fatou's Lemma*** *If f_j is a sequence of non-negative measurable functions (don't have to be increasing) then*

$$\int_E \liminf f_j d\mu \leq \liminf_j \int_E f_j d\mu$$

Proof. Set $g_k(x) = \inf_{j \geq k} f_j(x)$, non-negative and measurable, but also increasing, i.e. $g_{k+1}(x) \geq g_k(x)$. So we can apply monotone convergence theorem

$$\lim_{k \to \infty} g_k(x) = \liminf_j f_j \Longrightarrow \int_E \liminf f_j d\mu = \lim_{k \to \infty} \int_E g_k d\mu$$

For each $j \geq k$, $g_k(x) = \inf_{j \geq k} f_j \leq f_j(x)$ implies that

$$\int_E g_k(x) d\mu \leq \int_E f_j d\mu, \qquad \forall k \Longrightarrow \int_E g_k(x) d\mu \leq \inf_{j \geq k} \int_E f_j d\mu$$

thus

$$\int_E \liminf f_j d\mu = \lim_{k \to \infty} \int_E g_k(x) d\mu \leq \liminf_j \int_E f_j d\mu$$

□

Theorem. ***Lebesgue Dominated Convergence*** *We have $(X, \mathcal{F}, \mu)$. If f_j is a sequence of measurable functions, which converge pointwise $f_j(x) \to f(x)$ AND $\exists g$ integrable, non-negative such that $|f_j| \leq g$, $\forall j$. Then*

$$\int_E \lim_{j \to \infty} f_j d\mu = \lim_{j \to \infty} \int_E f_j d\mu$$

Proof. There is a trick. Apply Fatou's Lemma to $g + f_j \geq 0$, measurable. Then

$$\int_E \liminf_j (g + f_j) d\mu \leq \liminf \int_E (g + f_j) d\mu$$

Since $f_j(x) \to f(x)$ pointwise, $\liminf_j (g + f_j) = g + f$.

$$\int g d\mu + \int f d\mu = \int (g + f) d\mu \leq \liminf_j \left[\int^E g d\mu + \int^E f_j d\mu \right]$$

and so

$$\int_E f d\mu \leq \liminf_j \int_E f_j d\mu$$

Similarly, apply Fatou's Lemma to $g - f_j \geq 0$, we get

$$\int_E (g - f) d\mu \leq \liminf \left[\int_E g d\mu - \int_E f_j d\mu \right] \implies - \int_E f d\mu \leq \liminf_j \left[- \int_E f_j d\mu \right]$$

We know that $\liminf(-a_j) = -\limsup(a_j)$, so

$$- \int_E f d\mu \leq - \limsup_j \int_E f_j d\mu \implies \int_E f d\mu \geq \limsup_j \int_E f_j d\mu.$$

combining the two inequalities that we got yields

$$\liminf_j \int_E f_j d\mu \geq \int_E f d\mu \geq \limsup_j \int_E f_j d\mu$$

□