

Theorem. *An operator $B \in \mathcal{B}(H)$ is compact, (i.e. $\overline{B(\{\|u\| \leq 1\})}$ is compact) if and only if there exists a sequence of finite rank operators, T_n such that $\|B - T_n\| \rightarrow 0$.*

Theorem described in the next page...

Proof. Consider the opposite of $\pi_{(N)}$,

$$\pi_N f = \sum_{j \leq N} \langle f, \varphi_j \rangle \varphi_j$$

is of finite rank. Also $\pi_N^2 = \pi_N$ and $Id = \pi_N + \pi_{(N)}$ and $\pi_N \pi_{(N)} = 0$.

We claim that if B is compact, then $\pi_N B = T_n \rightarrow B$ in norm as $N \rightarrow \infty$. $B - T_N = B - \pi_N B = (Id - \pi_N)B = \pi_{(N)}B$.

By compactness of $\overline{B(\{u; \|u\| \leq 1\})}$, if $f \in \overline{B(\{u; \|u\| \leq 1\})}$ we know $\|\pi_{(N)}f\| < \epsilon$ if N large uniformly $\forall f \in \overline{B(\{u; \|u\| \leq 1\})}$. So $\forall u \in H, \|u\| = 1, \|\pi_{(N)}Bu\| < \epsilon$. So $\forall u \in H, \|\pi_{(N)}Bu\| < \epsilon\|u\| \Rightarrow \|\pi_{(N)}B\| \leq \epsilon$. (We have the converse left to prove). \square

Note about finite rank operators.

By definition, $B \in \mathcal{B}(H)$ is of finite rank if $B(H) = V$ is finite dimensional. The following lemma is very important.

Lemma. B finite rank $\Leftrightarrow \exists u_i, v_i \in H, i = 1, \dots, N$ such that

$$(*) \quad Bf = \sum_{i=1}^N \langle f, u_i \rangle v_i$$

Proof. Clear that $(*)$ implies $B(H) \subset \text{span}\{v_1, \dots, v_n\}$.

Conversely suppose $B(H) = V$ is finite dimensional. Choose an orthonormal basis of $V, \varphi_1, \dots, \varphi_N$. $\forall f \in H$ consider

$$F_i : H \ni f \mapsto \langle Bf, \varphi_i \rangle \in \mathbb{C}$$

We can see that this is bounded and thus continuous

$$|F_i(f)| = |\langle Bf, \varphi_i \rangle| \leq \|Bf\| \|\varphi_i\| \leq \|B\| \|f\|$$

Then F_i is a continuous linear functional, then Riesz implies that $F_i(f) = \langle f, u_i \rangle$ for some $u_i \in H$.

However, $Bf \in V$, so because φ_i is complete and orthonormal

$$Bf = \sum_{i=1}^N \langle Bf, \varphi_i \rangle \varphi_i = \sum_{i=1}^N F_i(f) \varphi_i = \sum_{i=1}^N \langle f, u_i \rangle \varphi_i$$

\square

Hence B of finite rank then B^* of finite rank, because

$$\langle Bu, v \rangle = \langle u, B^*v \rangle, \quad \forall u, v \in H$$

So then

$$\begin{aligned} \sum_{i=1}^N \langle u, u_i \rangle \langle v_i, v \rangle &= \langle u, B^* v \rangle \quad \forall u, v \in H \\ \Rightarrow B^* v &= \sum_{i=1}^N \langle v, v_i \rangle u_i \end{aligned}$$

So we can fully justify the statement above that B compact implies that B^* compact.

So we can also justify $\mathcal{C}(H)$ is a two-sided ideal. Because it is easy to see that $B_1 B_2$ is compact if B_1 is compact. To show that $B_2 B_1$ is compact if B_2 is compact, just show that $B_2 B_1 = (B_1^* B_2^*)^*$, which is compact, because the right is compact.

Theorem. *If $A \in \mathcal{C}(H)$ is a compact operator and $A^* = A$ (self-adjoint) then \exists a complete orthonormal basis, $\{\varphi_i\}$ such that each φ_i is an eigenvector, $A\varphi_i = \lambda\varphi_i$, $\forall i, \lambda_i \in \mathbb{R}$. We can also show*

1. *The set of all λ_i which does this is discrete and can only accumulate at 0*
2. *And for every $\lambda \neq 0$, $\dim\{u \in H; Au = \lambda u\} < \infty$.*

Proof. First Step Find a non-trivial eigenvalue/eigenvector (if $A \neq 0$).

Consider the function

$$\{u \in H; \|u\| = 1\} \ni u \mapsto \langle Au, u \rangle \in \mathbb{R}$$

($\overline{\langle Au, u \rangle} = \langle u, Au \rangle = \langle A^* u, u \rangle = \langle Au, u \rangle$, so $\langle Au, u \rangle$ is in fact real). Suppose $\langle Au, u \rangle$ assumes a positive value. Define

$$\sup_{u \in H, \|u\|=1} \langle Au, u \rangle = C \leq \|Au\| \|u\| \leq \|A\|$$

Then by definition of sup there exists a sequence $u_j \in H$, $\|u_j\| = 1$ such that $\langle Au_j, u_j \rangle \rightarrow C$.

By diagonalization choose a subsequence u_{j_n} such that $\pi_N u_{j_n} \rightarrow \pi_N u$ for some $u \in H$, $\forall N$ (weak convergence). Note: the previous is equivalent to $\langle u_{j_k}, v \rangle \rightarrow \langle u, v \rangle$, $\forall v \in H$.

Rename u_{j_n} as u_j .

We claim that $\langle Au, u \rangle = C$, i.e. the supremum is attained. This is true because compactness of A implies $Au_j \rightarrow Au$ (in norm). Why? We just prove that Au_j is Cauchy.

$$\begin{aligned} \|Au_n - Au_m\|^2 &= \|\pi_N(Au_n - Au_m)\|^2 + \|\pi_{(N)}(Au_n - Au_m)\|^2 \\ &\leq \|\pi_N(Au_n - Au_m)\|^2 + \epsilon^2 \quad \text{by choice of } N \text{ large} \\ &\leq \epsilon^2 + \epsilon^2 \quad \text{by choice of } n, m \text{ by weak convergence} \end{aligned}$$

By completeness $Au_j \rightarrow w$. And we know

$$\langle Au_j, v \rangle = \langle u_j, Av \rangle \rightarrow \langle u, Av \rangle = \langle Au, v \rangle$$

so $w = Au$.

So all we've done is extract a weakly convergent sequence, $Au_j \rightarrow Au$ and $u_j \rightarrow u$ (\rightarrow means weak convergence)

To show $\langle Au_j, u_j \rangle \rightarrow \langle Au, u \rangle$,

$$\begin{aligned} |\langle Au_j, u_j \rangle - \langle Au, u \rangle| &= |\langle Au_j - Au, u_j \rangle| + |\langle Au, u_j - u \rangle| \\ &\leq \|A_j - A\| \|u\| \end{aligned}$$

The 1st term converges because of strong convergence of A_j , and second term because $u_j \rightarrow u$ and Au fixed. So the supremum of $\langle Au, u \rangle$ on $\|u\| = 1$ is attained.

Now we have found u such that $\|u\| = 1$ and $\langle Au, u \rangle = \sup_{\|v\|=1} \langle Av, v \rangle$.

Step 2 Claim: u is an eigenvector for A .

Take $\varphi \in H$, $\|\varphi\| = 1$, with $\varphi \perp u$. Consider $u + t\varphi \in H$.

$$\|u + t\varphi\|^2 = \|u\|^2 + t^2\|\varphi\|^2 = 1 + t^2$$

Then

$$\left\| \frac{u + t\varphi}{(1 + t^2)^{1/2}} \right\| = 1$$

and this is "a little curve on the unit sphere" (parametrically defined), call it u_t . Then

$$\langle Au_t, u_t \rangle = \frac{1}{(1 + t^2)} \langle A(u + t\varphi), u + t\varphi \rangle = \frac{1}{(1 + t^2)} [\langle Au, u \rangle + t\langle Au, \varphi \rangle + t\langle A\varphi, u \rangle + t^2\langle A\varphi, \varphi \rangle]$$

This is a differentiable function, because it is a polynomial in t .

Also $\langle Au_t, u_t \rangle \leq \langle Au, u \rangle$, so u_t has a local max at $t = 0$. So

$$\left. \frac{d}{dt} \langle A(u + t\varphi), u + t\varphi \rangle \right|_{t=0} = \langle Au, \varphi \rangle + \langle A\varphi, u \rangle = 0$$

We can also consider $u + it\varphi$, because $\|i\varphi\| = 1$, $\langle i\varphi, u \rangle = 0$. The same argument implies that

$$\left. \frac{d}{dt} \langle A(u + it\varphi), u + it\varphi \rangle \right|_{t=0} = \langle Au, \varphi \rangle - \langle A\varphi, u \rangle = 0$$

So when we combine the above two statements we get

$$\begin{aligned} \langle Au, \varphi \rangle + \langle A\varphi, u \rangle &= 0 \\ \langle Au, \varphi \rangle - \langle A\varphi, u \rangle &= 0 \end{aligned}$$

So then $\langle Au, \varphi \rangle = 0$. Thus we have shown that

$$\boxed{\varphi \perp u \implies \varphi \perp Au}$$

The above is only true if $Au = \xi u$ for some $\xi \in \mathbb{C}$, because if $\varphi \perp u$ and $\varphi \perp Au$, then u and Au have to be contained in the same 1-dimensional subspace. Now, because $\langle Au, u \rangle$ is real, $\xi \in \mathbb{R}^+$. So indeed u is an eigenvector with $Au = \lambda u$, $\lambda > 0$.

Step 3: Call $u = u_1$. Consider the subspace of H ,

$$H_1 = \{v \in H; v \perp u_1\}$$

This is a Hilbert space with the same induced norm and

$$AH_1 \subset H_1$$

because $v \in H_1$, $v \perp u_1$ implies that $\langle Av, u_1 \rangle = \langle v, Au_1 \rangle = 0$ and so $\lambda \langle v, u_1 \rangle = 0$.

Also $A = A^*$ is compact. We can repeat the process.

We get an orthonormal sequence $\varphi_1, \dots, \varphi_N \dots$. And

1. If it terminates, then $\langle Au, u \rangle = 0$, $\forall u \perp \varphi_1, \dots, \varphi_N$
2. If it does not terminate we claim that $\langle Au, u \rangle = 0$, $\forall u \perp \varphi_j$, $\forall j \in \mathbb{N}$.

We state this formally

Lemma. *2 above holds, and furthermore if we have such a sequence $\varphi_1, \varphi_2, \dots$ with eigenvalues $\lambda_1, \lambda_2, \dots$, then $|\lambda_j| \rightarrow 0$ as $j \rightarrow \infty$.*

Proof. **NB** $|\lambda_j|$ is decreasing (non-increasing) by construction, since $|\lambda_j| = \sup_{\|u\|=1} |\langle Au, u \rangle|$.

Now, if it is not true that $|\lambda_j| \rightarrow 0$, i.e. $\exists c > 0$ such that $|\lambda_j| \geq c$, $\forall j$. Then because $|\lambda_j|$ is increasing, this implies that we can choose a subsequence with $\lambda_{j_i} > 0$ or $\lambda_{j_i} < 0$, on a subset of $\{\varphi_{j_i}\}_i$ of $\{\varphi_n\}_n$.

So then, by the way we have constructed the eigenvalues, $|\langle Au, u \rangle| > c$ on $\text{span}\{\varphi_{j_1}, \dots, \varphi_{j_n}, \dots\} = G$. But this contradicts the fact that $|A\varphi_j| \rightarrow 0$ as $j \rightarrow \infty$ (which follows from the compactness of A) □

So $\langle Au, u \rangle = 0$, $\forall u \perp \varphi_1, \dots, \varphi_n, \dots$ □

Now, the above result (in particular the last line) implies that

$$H = H_1 \oplus H_2$$

where

$$\begin{aligned} H_1 &= \overline{\text{span}\{\varphi_1, \varphi_2, \dots\}} \\ H_2 &= \{u; u \perp \varphi_j, \forall j\} \end{aligned}$$

Because if $u \in H_1$ then

$$u_1 = \sum_{j=0}^{\infty} \langle u, \varphi_j \rangle \varphi_j \implies u - u_j \perp \varphi_j \quad \forall j$$

We know that $\langle Au, u \rangle = 0$ on H_2 . Note that $A : H_2 \rightarrow H_2$, since

$$u \perp \varphi_j \implies \langle Au, \varphi_j \rangle = \langle u, A\varphi_j \rangle = \lambda_j \langle u, \varphi_j \rangle = 0$$

So in fact $A = 0$ on all of H_2 , because $\forall u, v \in H_2$

$$\begin{aligned}\langle A(u+v), u+v \rangle &= 0 = \langle Au, v \rangle + \langle v, Au \rangle \\ \langle A(u+iv), u+iv \rangle &= 0 = -\langle Au, v \rangle + \langle v, Au \rangle\end{aligned}$$

then $\langle Au, v \rangle = 0$ implies that $A = 0$.

We can get a corollary of the above for normal operators.

Definition. An operator is **normal** if

$$[A^*, A] = A^*A - AA^* = 0$$

This is slightly weaker than $A = A^*$, it just says that A and A^* commute.

Corollary. If $A \in \mathcal{C}(H)$ and A normal, then there exists a complete orthonormal basis of eigenvectors of A . The only difference between this and the above is that $\lambda_j \in \mathbb{C}$, but still $|\lambda_j| \rightarrow 0$.

Proof. $A \in \mathcal{C}(H)$ then $A^* \in \mathcal{C}(H)$ which implies that

$$\frac{A + A^*}{2} = B \in \mathcal{C}(H), \quad B^* = B$$

apply our theorem above to B . So there exists an orthonormal basis of eigenvectors for B , with $B\varphi_j = \lambda_j\varphi_j$, $\lambda_j \in \mathbb{R}$, $|\lambda_j| \rightarrow 0$.

We can also define

$$D = \frac{A - A^*}{i} \in \mathcal{C}(H), \quad D = D^*$$

Now $[A, A^*] = 0$ implies that $[D, B] = 0$.

For each $0 \neq \lambda \in \mathbb{R}$ define

$$H_\lambda = \{u \in H; Bu = \lambda u\}$$

this is finite dimensional (trivially since there can only be a finite number of eigenvalues that are λ , or else $|\lambda_j| \rightarrow 0$ would not be true). By commutivity if $u \in H_\lambda$, then $B(Du) = DBu = D\lambda u = \lambda Du$, so in fact $D : H_\lambda \rightarrow H_\lambda$. So we can choose an orthonormal basis of H_λ such that $D\varphi_j^\lambda = \eta_j^\lambda\varphi_j^\lambda$.

So for A , we get the following eigenvectors and eigenvalues

$$A = B + iD : H_\lambda \rightarrow H_\lambda, \quad A\varphi_j^\lambda = (\lambda + i\eta_j^\lambda)\varphi_j^\lambda$$

We can do this for every λ which is an eigenvalue of B , and we get a countable complete orthonormal basis. Note that on $H_2(B) = \{Bu = 0\}$ we can just choose eigenvectors/values from D . \square