Practice Final

This exam is closed book, no books, papers or recording devices permitted. You may use theorems from class, or the book, provided you can recall them correctly.

Problem 1

Suppose $f \in L^1([0,1])$ and $\int_{[0,1]} f\chi dx = 0$ for all simple measurable functions χ on $[0,1]$. Show thatt $f = 0$ almost everywhere with respect to Lebesgue measure.

Ans. Take χ to be the characteristic function of the measureable set $\{x \in [0,1]; f(x) > 0\}$. Then $\int_{[0,1]} f\chi dx = \int_{[0,1]} f^+ dx = 0$ implies that $f^+ = 0$ a.e. - the same argument works for f^- so $f = 0$ a.e.

Problem 2

Suppose A is a compact operator on a Hilbert space H, and that A^*A has no positive eigenvalues, show that $A = 0$.

Ans. Since A^*A is compact and selfadjoint there is a complete orthonormal basis of its eigenvectors. All the eigenvectors must be non-negative, since $A^*Au = \lambda u$ imples $\|Au\}^2 = \lambda\|u\|^2$. So, if there are no positive eigenvalues they must all be zero. Thus $A^*Au = 0$ for all u implies $\|Au\| = 0$ for all u, so $A = 0$.

Problem 3

Give an example of a function $u \in L^2(\mathbb{R})$ which is continuous but is such that its Fourier transform $\hat{u} \notin L^1(\mathbb{R})$.

Ans. For any N it is easy to find a non-negative continuous function, g, with maximum N supported in $[0, N^{-4}]$. The integral of its square is then less than $1/N^2$. Consider the function

$$f(x) = \sum_{N \geq 1} g(x - N).$$

All the terms have disjoint supports and f is square integrable. If its Fourier transform was in L^1 the function f would have to be continuous, which it is, and would also have to vanish infinity, which it does not. [This one is a bit tricky.]

Problem 4

Suppose $u \in L^2([-\pi,\pi])$ and there exists $v \in L^2([-\pi,\pi])$ such that

$$\int_{[-\pi,\pi]} u(x)\frac{d}{dx}\phi(x)dx = \int_{[-\pi,pi]} v(x)\phi(x)dx$$

for all smooth 2π-periodic functions, ϕ, on the real line. Show that u has a continuous representative in $L^2([-\pi,\pi])$.

Ans. Plug e^{ikx} into the identity for each $k \in \mathbb{N}$ and you find that the Fourier coefficients c_k of u satisfy $ikc_k = d_k$ where d_k are the Fourier coefficients of $v \in L^2([-\pi,\pi])$. Thus

$$\sum_k k^2|c_k|^2 < \infty.$$

This is enough to imply that the Fourier series for u converges uniformly so f \`is' continuous - has a representative which is continuous.

Problem 5

Suppose $f \in \mathcal{S}(\mathbb{R})$ has Fourier transform satisfying $\hat{f}(\xi) = 0$ in $|\xi| < 1$. Show that there exists $g \in \mathcal{S}(\mathbb{R})$ such that $f(x) = \frac{d^2}{dx^2}g(x)$ for all $x \in \mathbb{R}$.

Ans. The function $h(\xi) = -\hat{f}(\xi)/\xi^2$ in $|\xi| > 1/2$, $h(\xi) = 0$ in $|\xi| \leq 1/2$ is in $\mathcal{S}(\mathbb{R})$ and satisfies $(-\xi)^2 h(\xi) = \hat{f}(\xi)$. Thus if g is the inverse Fourier transform of h it is in $\mathcal{S}$ and satisfies $\frac{d^2}{dx^2}g = f$.

Problem 6

Show that there is no element of $L^1([-\pi,\pi])$ satisfying

$$\int_{[-\pi,\pi]} f(x)e^{ik^3x} = 1 \ \forall \ k \in \mathbb{N}.$$

Ans. This stops the Fourier coefficietns of f from vanishing at ∞.

Problem 7

Suppose $f \in L^2([-\pi,\pi]))$ had Fourier coefficients $c_j,\ j \in \mathbb{Z}$ satisfying

$$\sum k \in \mathbb{Z} k|c_k| < \infty.$$

Show that there exists a function $g \in L^2([-\pi,\pi])$ such that

$$\int_{[-\pi,pi]} g(x)\phi(x)dx = \int_{[-\pi,pi]} f(x)\left(\frac{d}{dx}\phi(x) + \cos(x)\phi(x)\right)dx$$

for all smooth 2π-periodic functions ϕ on the real line.

Ans. The given condition implies he uniform convergence of the (formal) Fourier series for df/dx. So f has a continuous first derivative. Integration by parts is then justified in the given identity so we can take $g = -\frac{df(x)}{dx} + f(x)\cos x$.

Problem 8

If $f \in C([0,1])$, show that

$$(Au)(x) = \int_x^1 f(t)u(t)dt,\ x \in [0,1],$$

defines a compact operator $A : L^2([0,1]) \longrightarrow L^2([0,1])$.

Ans. Change variables to shift the integral to $[-\pi,\pi]$ (or use Fourier series on $[0,1]$) by setting $y = 2\pi x - \pi$. The

$$Au'(y) = \frac{1}{2\pi}\int_y^\pi f(s/2\pi - 1)u'(s)ds.$$

Anyway, we get the same sort of integral with a different continuous function. Now, integrating agains e^{ikx} we again find that the Fourier coefficients a_k of Au' satisfy $\sum_k k^2|a_k|^2 < C$ whenever $\|u'\| \leq 1$. This implies that A maps the ball into a compact set.

Problem 9

Show that there is an infinite orthonormal sequence $u_j \in L^2(\mathbb{R})$ with each element satifying

$$\widehat{u_j} = c_j u_j,\ c_j \in \mathbb{C}.$$

Ans. The eigenfunctions of the harmonic oscillator.