First test, solutions

You are permitted to bring the book `Adams and Guillemin: Measure Theory And Probability' with you - just the book, nothing else is permitted (and no notes in your book!) You may use theorems, lemmas and propositions from the book.

1. Consider the sequence of functions $f_n(x) = (x + n^{-1})^{-\frac{1}{4}}$ on $[0,1]$ for $n = 1, 2, \ldots$.

   Explain why these functions are measurable with respect to Lebesgue measure. Assuming the equality of the Lebesgue and Riemann integrals for continuous functions what can you deduce, and how, about the Lebesgue integrability of $f(x) = x^{-\frac{1}{4}}$ on $[0,1]$?

   Solution:- The $f_n \geq 0$ are continuous, hence measurable (or check this directly). Moverover $f_n(x)$ increases with n so the monotone convergence theorem applies and shows that the limit $f(x)$ is measurable and has integral

   $$\int_{[0,1]} f(x)dx = \lim_{n\to\infty} \int_{[0,1]} f_n(x)dx.$$

   By the equality of Riemann and Lebesgue integrals for continuous functions we can compute the integrals of the f_n :

   $$\int_{[0,1]} f_n(x)dx = \frac{4}{3}(x + \frac{1}{n})^{\frac{3}{4}}|_0^1 \to \frac{4}{3}.$$

   Thus f is Lebesgue integrable with integral $4/3$.

   Note: The limit f is not bounded, hence not Riemann integrable in the strict sense (it has an improper Riemann integral).

2. Recall that a collection of subsets of a given set is a σ-ring if the difference of any two elements is an element and it contains the union of any countable collection of its elements. Given any collection C of subsets of a set X show that there is a smallest σ-ring of sets containing C.

Solution:- The collection of all subsets, 2^X, if X is a σ-ring containing $\mathcal{C}$. Consider the non-empty intersection

$$\mathcal{R} = \bigcap_{\mathcal{A} \text{ a } \sigma\text{-ring with } \mathcal{C} \subset \mathcal{A}} \mathcal{A}.$$

Then $\mathcal{R}$ us closed under differences and countable unions, since each of the collections $\mathcal{A}$ is so closed and it contains $\mathcal{C}$. Moreover, by definition, any σ-ring containing $\mathcal{C}$ must be one of the $\mathcal{A}$ so must contain $\mathcal{R}$, which is therefore the smallest.

Constructing $\mathcal{R}$ directly is more painful.

3. Briefly recall the manner in which a countably additive measure $\mu : \mathcal{R} \longrightarrow [0, \infty)$ on a ring of subsets of a set X is completed to a countably additive measure on a σ-ring (μ) of subsets of X. Show that if $\mu_i : \mathcal{R} \longrightarrow [0, \infty)$, for $i = 1, 2$, are two such countably additive finite measures on a ring $\mathcal{R}$ of subsets of X and $\mu_1(R) \leq \mu_2(R)$ for all $R \in \mathcal{R}$ then $\mathcal{M}(\mu_1) \supset \mathcal{M}(\mu_2)$ and that the extended measures satisfy $\mu_1(M) \leq \mu_2(M)$ for all $M \in \mathcal{M}(\mu_2)$.

Solution:- Given a finite, but countably additive, measure, μ, on $\mathcal{R}$ we first define the outer measure $\mu^*(A)$ for any set A as the infimum of $\sum_j \mu(A_j)$ where the $A_j \in \mathcal{R}$ cover A (or as $+\infty$ if no such cover exists). Thus if $\mu_1 \leq \mu_2$ are two such measures on $\mathcal{R}$ then $\mu_1^*(A) \leq \mu_2^*(A)$ for all subsets A. Then we define $\mathcal{M}(\mu)$ as the collection of subsets which are μ^*-approximable by elements of $\mathcal{R}$, that is $A \in \mathcal{M}_F(\mu)$ if and only if there exists a sequence A_j in $\mathcal{R}$ with $\mu^*(S(A, A_j)) \to 0$. If A is μ_2^* approximable it is certainly μ_1^* approximable, since $\mu_1^* \leq \mu_2^*$. Thus $\mathcal{M}_F(\mu_1) \supset \mathcal{M}_F(\mu_2)$. The measurable sets, forming $\mathcal{M}(\mu)$, are by definition just the countable unions of elements of $\mathcal{M}_F(\mu)$ so again $\mathcal{M}(\mu_1) \supset \mathcal{M}(\mu_2)$. On $\mathcal{M}(\mu)$ the extended measure is just μ^* so necessarily $\mu_1 \leq \mu_2$ on $\mathcal{M}(\mu_2)$ where they are both defined.

4. Suppose that $0 \leq f(x) \leq R$ is a bounded measurable function on $[0, 1]$, explain what its Lebesgue integral is and why it is finite.

Solution:- For a non-negative measurable function the integral is

$$\int_{[0,1]} f dx = \sup\{I(s); 0 \leq s \leq f \text{ with } s \text{ simple measurable }\}.$$

For simple functions $s_1 \leq s_2$ implies $I(s_1) \leq I(s_2)$ and since $f \leq R$ we know that $s \leq f$ implies $s \leq R$ so $I(s) \leq R$ and the supremum on the right is finite, hence so is the integral.

5. Let $f_n : [0,1] \longrightarrow [0,\infty)$ be an increasing sequence of functions on $[0,1]$ ($f_{n+1}(x) \geq f_n(x)$ for all $x \in [0,1]$ and all n) which are Lebesgue measurable with $\int_{[0,1]} f_n dx$ finite for each n but unbounded as $n \to \infty$. Show that, given $\epsilon > 0$ there is a measurable subset $A \subset [0,1]$ with $\mu_{\text{Leb}}(A) < \epsilon$ such that $\int_A s_n dx \to \infty$.

Solution:- Given $\epsilon > 0$ divide the interval into $N > 1/\epsilon$ equal subintervals, A_i, each of which has measure $< \epsilon$. If $\int_{A_i} f_n ds$ is bounded for each i then so is their sum, which is $\int_{[0,1]} f_n dx$. Thus for at least one interval the sequence is unbounded. Since the sequence of functions is increasing, so is this sequence, hence it tends to ∞.