Theorem. *$\varphi_0, \varphi_1, \ldots$ form a complete set in $L^2(\mathbb{R})$ (scrap the proof from last time).*

Proof. We use complex variables. Show that if $f \in L^2(\mathbb{R})$ and $\langle f, \varphi_j \rangle = 0\ \forall j \in \mathbb{N}$ then $f = 0 a.e.$

Consider the Fourier transform of $F(x) = f(x)e^{-x^2/2} = f(x)\varphi_0$. $F(x) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, because by Cauchy-Schwarz

$$\int |f(x)e^{-x^2/2}| \leq \left(\int |f|^2\right)^{1/2} \left(\int e^{-x^2/2}\right)^{1/2} < \infty$$

$\widehat{F}(s) = \int F(x)e^{-ixs}dx$ exists in $C(\mathbb{R})$. In fact the integral exists for all $s \in \mathbb{C}$,

$$\left| \int_{\mathbb{R}} f(x)e^{-x^2/2-ixs}dx \right| \leq \int_{\mathbb{R}} |f| e^{-x^2/2+|x||s|}dx$$

Apply Cauchy-Schwarz again and we get the above bounded by

$$\leq \left(\int_{\mathbb{R}} |f|^2\right)^{1/2} \left(\int_{\mathbb{R}} e^{-x^2+2|x||s|}\right)^{1/2}$$

and

$$\int_{\mathbb{R}} e^{-x^2+2|x||s|}dx \leq 2\int_{-\infty}^{\infty} e^{-x^2+|x||s|}dx = e^{|s|^2/4} \int_{-\infty}^{\infty} e^{-(x+\frac{|s|}{2})^2}dx \leq 2\sqrt{\pi}e^{\frac{|s|^2}{4}}$$

so the integral exists for all complex c.

Now we want to show that $\widehat{F}(s)$ is **entire** if $s \in \mathbb{C}$, i.e. it is the sum of its Taylor series at $s = 0$.

We claim that

1. That the taylor series at 0 for $\widehat{F}(s)$ converges $\forall s \in \mathbb{C}$.
2. Formally,
$$\widehat{F}^{(n)}(0) = \int_{\mathbb{R}} f(x)e^{-x^2/2}(-1)^n x^n = (-1)^n \langle f, x^n e^{-x^2/2} \rangle = 0$$
where the final equality comes from the assumption f should be orthogonal to things of the form $p_n e^{-x^2/2}$

An outline of our entire argument is $\widehat{F} \equiv 0$ **then** $F(x) = 0$ **in** $L^2(\mathbb{R}) \Rightarrow f(x)e^{-x^2/2} = 0$ **almost everywhere, implies** $f = 0 \in L^2(\mathbb{R})$ **which implies that** $\{\varphi_i\}$ **is complete**

We prove the two things above. First plug in the Taylor series for e^{-isx}, then we get

$$e^{-isx} = \sum_{n=0}^{N-1} \frac{(-isx)^n}{n!} + R_N(x,s)$$

so

$$\widehat{F}(s) = \sum_{n=0}^{N-1} \frac{(-i)^n s^n}{n!} \underbrace{\int_{\mathbb{R}} f(x)e^{-x^2/2}x^n dx}_{\langle f, x^n e^{-x^2/2}\rangle} + \int_{\mathbb{R}} f(x)e^{-x^2/2} R_N(x,s)dx$$

we want to prove that the remainder term at 0 decreases quickly. Using the Legendre form for the remainder we get

$$\left| f(1) - \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} \right|$$

so with our particular case with $f(t) = e^{tz}$ we have

$$\left| e^z - \sum_{n=0}^{N-1} \frac{z^n}{n!} \right| \leq \frac{1}{(N-1)!} \int_0^1 |s^{N-1} z^N e^{tzs}| ds$$

$$|r_N(z)| \leq \frac{e^{|z|}}{(N-1)!} |z|^n$$

in our original equation $R_N(x,s) = r_N(-isx)$ and so our remainder term when $z = xs$ becomes (and then we apply Cauchy Schwartz)

$$\frac{|s|^N}{(N-1)!} \int_{\mathbb{R}} f(z) e^{-x^2/2} e^{|x||s|} x^N dx \leq \left(\int |f|^2 dx \right)^{1/2} \left(\int x^{2N} e^{-x^2 + 2|x||s|} \right)^{1/2} \leq \frac{C(N!)^{1/2}}{(N-1)!}$$

We get the last bound because we can estimate using integration by parts (brings down the N's) (kind of like the Γ-function). So we have in fact proved that

$$\widehat{F}(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} \widehat{F}^{(n)}(0)$$

so in fact $\varphi_0, \varphi_1, \ldots$ is complete. □

10 Sobolev Spaces

Let $\mathbb{S}$ be the circle, that is $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$ is the equivalence class $p = y$ if $p - y \equiv 0 \mod 2\pi\mathbb{Z}$. A circle in the unique 1-dimensional bounded compact manifold.

So functions on $\mathbb{S}$ can be identified with 2π periodic functions on $\mathbb{R}$. For L^1 and L^2 we have a simple identification using Fourier series. Because when constructing Fourier series, we did it by thinking about $L^2([-\pi,\pi])$ and $L^1([-\pi,\pi])$, so

$$L^2([-\pi,\pi]) \cong L^2(\mathbb{S}) \cong \{f : \mathbb{R} \to \mathbb{C}, \text{period } 2\pi\} \qquad L^1([-\pi,\pi]) \cong L^1(\mathbb{S})$$

Fourier Series: $f \in L^2(\mathbb{S}) \cong L^2([-\pi,\pi])$ then f is the limit of its Fourier series in $L^2(\mathbb{S})$

Definition. We define $C^\infty(\mathbb{S})$ as follows

$$C^\infty(\mathbb{S}) = \{u : \mathbb{R} \to \mathbb{C}, u \text{ infinitely differentiable, period } 2\pi\}$$

Then we conclude that $C^\infty(\mathbb{S}) \hookrightarrow L^2(\mathbb{S})$ is dense (in the L^2 topology). Just a consequence of fourier series, and the fact that $c_n e^{inz} \in C^\infty(\mathbb{S})$.

Definition. Integration on $\mathbb{S}$ We define integration, as would be obvious, by

$$\int_{\mathbb{S}} f dx = \int_{-\pi}^{\pi} f dx, \qquad f \in L^1(\mathbb{S}) \cong L^1([-\pi,\pi])$$

Theorem. ***Integration by Parts*** *Integration by parts works on* $\mathbb{S}$*, i.e. for* $f, g \in C^\infty(\mathbb{S})$ *then*

$$\int_{\mathbb{S}} f \frac{dg}{dx} dx = -\int_{\mathbb{S}} \frac{df}{dx} g dx$$

Proof. Follows from 2π periodicity

$$\int_{-\pi}^{\pi} f \frac{dg}{dx} dx = \int_{-\pi}^{\pi} \frac{d}{dx}(fg) dx - \int_{-\pi}^{\pi} \frac{df}{dx} g dx = -\int_{\mathbb{S}} \frac{df}{dx} g dx + fg\Big|_{-\pi}^{\pi} = -\int_{\mathbb{S}} \frac{df}{dx} g dx$$

□

We don't really have a formulation of a derivative of any sort, but we can define these:

Definition. For $f \in L^2(\mathbb{S})$ we say that

$$\frac{df}{dx} = h \in L^2$$

"weakly" if $\exists h \in L^2(\mathbb{S})$ such that

$$\int_{\mathbb{S}} f \frac{d\varphi}{dx} dx = -\int_{\mathbb{S}} h\varphi dx, \qquad \forall \varphi \in C^\infty(\mathbb{S})$$

This is just an application of integration by parts. Note if f is smooth this is true. We also have a strong definition of derivatives

Definition. For $f \in L^2(\mathbb{S})$, h is the strong derivative of f if $\exists f_n \in C^\infty(\mathbb{S})$ such that $f_n \to f$ in $L^2(\mathbb{S})$ and $\frac{df_n}{dx} \to h$ in the $L^2(\mathbb{S})$ norm.

Theorem. *h is unique if it exists.*

Proof. If we have two derivatives h_1 and h_2 then

$$-\int h_1\varphi = -\int h_2\varphi, \qquad \forall\varphi \in C^\infty(\mathbb{S})$$

but $C^\infty(\mathbb{S})$ is dense in $L^2(\mathbb{S})$ so

$$-\int (h_1 - h_2)\varphi dx = 0 \Longrightarrow h_1 - h_2 = 0 \ a.e.$$

□

Theorem. *Strong derivatives imply existence of a weak derivative.*

Proof. If $\varphi \in C^\infty(\mathbb{S})$ and h is the strong derivative of $f \in L^2(\mathbb{S})$ then

$$\int_{\mathbb{S}} f\frac{d\varphi}{dx}dx = \lim_{n\to\infty}\int_{\mathbb{S}} f_n\frac{d\varphi}{dx}dx = \lim_{n\to\infty} -\int \frac{df_n}{dx}\varphi = -\int_{\mathbb{S}} h\varphi dx$$

because $\frac{df_n}{dx}$ converges in L^2. □

Using Fourier series we can see that

Theorem. *If $f \in L^2(\mathbb{R})$ then f has a strong derivative in L^2 if and only if it has a weak derivative in $L^2(\mathbb{S})$ if and only if the Fourier series*

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} inc_n e^{inx}, \qquad c_n = \int_{\mathbb{S}} fe^{-inx}dx$$

converges in $L^2(\mathbb{S})$. NOTE: Convergence of the above in L^2 happens if and only if $\sum_{n\in\mathbb{Z}} n^2|c_n|^2 < \infty$.

Proof. If f has a weak derivative in $L^2(\mathbb{S})$ then $\exists h \in L^2(\mathbb{S})$ such that

$$\int_{\mathbb{S}} f\frac{d\varphi}{dx} = -\int_{\mathbb{S}} h\varphi \qquad \forall\varphi \in C^\infty(\mathbb{S})$$

If we set $\varphi = e^{-inx}$ then the above becomes

$$-in\int_{\mathbb{S}} fe^{-inx}dx = -\int_{\mathbb{S}} he^{-inx}dx = -d_n$$

where d_n are the nth fourier coefficient of the derivative. Then we see that $inc_n = d_n$, $\forall n$.

But $h \in L^2(\mathbb{S})$ then

$$\sum_{n\in\mathbb{Z}} |d_n|^2 < \infty \Longrightarrow \sum_{n\in\mathbb{Z}} n^2|c_n|^2 < \infty$$

existence of the weak derivative implies that

$$\sum_{n\in\mathbb{Z}} n^2|c_n|^2 < \infty \Longrightarrow h = \sum_{n\in\mathbb{Z}} inc_n e^{inx}dx, \text{ in } L^2$$

now h is a strong derivative of f since

$$f_N = \frac{1}{2\pi}\sum_{|n|\leq N} c_n e^{inx} \to f, \text{in } L^2(\mathbb{S})$$

and $f_N \in C^\infty(\mathbb{S})$ and because this is a finite sum we can differentiate term by term

$$\frac{df_N}{dx} = \frac{1}{2\pi}\sum_{|n|\leq N} inc_n e^{inx} \to h, \quad \text{in } L^2(\mathbb{S})$$

So there is no ambiguity in saying $f \in L^2(\mathbb{S})$ has a derivative $df/dx \in L^2(\mathbb{S})$. □

Definition. Sobolev Space

$$H^1(\mathbb{S}) = \left\{ f \in L^2(\mathbb{S}) : \frac{df}{dx} \in L^2(\mathbb{S}) \text{ weakly/strong} \right\}$$

and so by the theorem above $f \in H^1(\mathbb{S})$ if and only if

$$f = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} c_n e^{inx}, \qquad \sum_{n\in\mathbb{Z}} n^2|c_n|^2 < \infty$$

Theorem. *$H^1(\mathbb{S})$ is a Hilbert space with the inner product*

$$\langle f, g\rangle_{H_1} = \left\langle \frac{df}{dx}, \frac{dg}{dx} \right\rangle + \langle f, g\rangle = \int_{\mathbb{S}} \frac{df}{dx}\overline{\frac{dg}{dx}} + \int_{\mathbb{S}} f\overline{g}$$

and the norm

$$\|f\|_{H^1}^2 = \int_{\mathbb{S}} \left|\frac{df}{dx}\right|^2 dx + \int_{\mathbb{S}} |f|^2 dx$$

The extra term just accounts for constant function, whose norm should not be 0.

Proof. This is clearly a norm. What about completeness? If $f_n \in H^1(\mathbb{S})$ is Cauchy then $f_n \to f$ in $H^1(\mathbb{S})$. If f_n is Cauchy, then df_n/dx and f_n are Cauchy in L^2 because

$$\left\| \frac{df_n}{dx} \right\|_{L^2} \leq \|f\|_{H^1}, \qquad \|f\|_{L^2} \leq \|f\|_{H^1}$$

so f_n converges to some f in $L^2(\mathbb{S})$ (by completeness of $L^2(\mathbb{S})$ and df_n/dx converges to some h in $L^2(\mathbb{S})$. But this is why we thought about the weak derivative in the first place, so

$$\int_{\mathbb{S}} f_n \frac{d\varphi}{dx} = -\int_{\mathbb{S}} \frac{df_n}{dx}\varphi, \qquad \forall n \forall \varphi \in C^\infty(\mathbb{S})$$

so

$$\int_{\mathbb{S}} f \frac{d\varphi}{dx} = -\int_{\mathbb{S}} h\varphi, \qquad \forall \varphi \in C^\infty(\mathbb{S})$$

so $h = \frac{df}{dx}$ for $f \in H^1(\mathbb{S})$. So $f_n \to f$ and $\frac{df_n}{dx} \to \frac{df}{dx}$ implies that $f_n \to f$ in $H^1(\mathbb{S})$ since

$$\|f\|_{H^1} \leq \|f\|_{L^2} + \left\|\frac{df}{dx}\right\|_{L^2}$$

$\square$