So in fact $A = 0$ on all of H_2, because $\forall u, v \in H_2$

$$\begin{aligned}\langle A(u+v), u+v\rangle =&0 = \langle Au, v\rangle + \langle v, Au\rangle \\ \langle A(u+iv), u+iv\rangle =&0 = -\langle Au, v\rangle + \langle v, Au\rangle\end{aligned}$$

then $\langle Au, v\rangle = 0$ implies that $A = 0$.

We can get a corollary of the above for normal operators.

Definition. An operator is **normal** if

$$[A^*, A] = A^*A - AA^* = 0$$

This is slightly weaker than $A = A^*$, it just says that A and A^* commute.

Corollary. *If $A \in \mathcal{C}(H)$ and A normal, then there exists a complete orthonormal basis of eigenvectors of A. The only difference between this and the above is that $\lambda_j \in \mathbb{C}$, but still $|\lambda_j| \to 0$.*

Proof. $A \in \mathcal{C}(H)$ then $A^* \in \mathcal{C}(H)$ which implies that

$$\frac{A + A^*}{2} = B \in \mathcal{C}(H), \qquad B^* = B$$

apply our theorem above to B. So there exists an orthonormal basis of eigenvectors for B, with $B\varphi_j = \lambda_j\varphi_j$, $\lambda_j \in \mathbb{R}$, $|\lambda_j| \to 0$.

We can also define

$$D = \frac{A - A^*}{i} \in \mathcal{C}(H), \qquad D = D^*$$

Now $[A, A^*] = 0$ implies that $[D, B] = 0$.

For each $0 \neq \lambda \in \mathbb{R}$ define

$$H_\lambda = \{u \in H; Bu = \lambda u\}$$

this is finite dimensional (trivially since there can only be a finite number of eigenvalues that are λ, or else $|\lambda_j| \to 0$ would not be true). By commutivity if $u \in H_\lambda$, then $B(Du) = DBu = D\lambda u = \lambda Du$, so in fact $D : H_\lambda \to H_\lambda$. So we can choose an orthonormal basis of H_λ such that $D\varphi_j^\lambda = \eta_j^\lambda\varphi_j^\lambda$.

So for A, we get the following eigenvectors and eigenvalues

$$A = B + iD : H_\lambda \to H_\lambda, \qquad A\varphi_j^\lambda = (\lambda + i\eta_j^\lambda)\varphi_j^\lambda$$

We can do this for every λ which is an eigenvalue of B, and we get a countable complete orthonormal basis. Note that on $H_2(B) = \{Bu = 0\}$ we can just choose eigenvectors/values from D. □

13.3 Examples

Example. Hilber-Schmidt Operators

$F \in L^2([-\pi,\pi]\times[-\pi,\pi])$. The following defines a compact operator:

$$A : L^2([-\pi,\pi]) \to L^2([-\pi,\pi]) \qquad Au(x) = \int_{[-\pi,\pi]} F(x,y)u(y)dy$$

For $\varphi \in L^2([-\pi,\pi])$ consider

$$\begin{aligned}\langle Au, \varphi\rangle &= \int \overline{\varphi(x)} \int F(x,y)u(y)dydx \\ &= \int_{[-\pi,\pi]}\int_{[-\pi,\pi]} F(x,y)\overline{\varphi(x)}u(y)dxdy \\ &= \int_{[-\pi,\pi]^2} F(x,y)\overline{\varphi(x)}u(y)dxdy\end{aligned}$$

Clearly $\overline{\varphi(x)}u(y) \in L^2([-\pi,\pi]^2)$, so

$$|\langle Au,\varphi\rangle| = \left|\int F(x,y)\overline{\varphi(x)}u(y)dxdy\right| \le \left(\int |F(x,y)^2dxdy\right)^{1/2}\left(\int |u(y)|^2|\varphi(x)|^2dxdy\right)^{1/2}$$

The above is less than $C\|u\|_2\|\varphi\|_2$, so

$$|\langle Au,\varphi\rangle| \le C\|u\|_2\|\varphi\|_2$$

Now fix $u \in L^2([-\pi,\pi])$ and define a function

$$L^2([-\pi,\pi]) = H \ni \varphi \mapsto \langle Au,\varphi\rangle$$

Now, this mapping is continuous since it is bounded. So we can apply the Riesz Representation Theorem and we can nicely define Au.

A is compact, because it is the norm limit of a finite rank operator. To prove this, we need to know the following.

Question. Can we use Fourier Series in both variables?

Well, define

$$\varphi_{kl} = \frac{1}{2\pi}e^{ikx}e^{ily} \in L^2([-\pi,\pi]^2) \qquad \forall k,l \in \mathbb{Z}$$

Then

$$\langle \varphi_{kl}, \varphi_{k'l'}\rangle = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(k-k')x}e^{(l-l')y}dxdy$$

is 0 if $k = k'$ OR $l = l'$. Essentially is 0 if $(k,l) \neq (k',l')$. So φ_{kl} is an orthonormal set on $L^2([-\pi,\pi]^2)$. This is complete as well (**Exercise: Prove it**).

By Bessel's inequality (regardless of completeness)

$$(\dagger)\ G(x,y) = \sum_{k,l} \langle F, \varphi_{kl} \rangle \varphi_{kl}$$

converges in $L^2([-\pi,\pi])$. And in fact

Lemma.

$$(*) \int G(x,y)\overline{\varphi(x)}u(y)dxdy = \int F(x,y)\overline{\varphi(y)}u(y)dxdy \qquad \forall u, \varphi \in L^2$$

Proof. Certainly (*) is true for $\varphi = \varphi_j$, $u = \varphi_k$, for any k, j

$$LHS = \int G(x,y)e^{-ijx}e^{iky} = \langle F, \varphi_{jk} \rangle = RHS$$

Since

$$\varphi = \sum \langle \varphi, \varphi_j \rangle \varphi_j \qquad u = \sum \langle u, \varphi_j \rangle \varphi_j$$

$LHS = RHS$ in general. So G gives the same operation as F. □

The fact that (†) converges in $L^2([-\pi,\pi]^2)$ implies that

$$\|A - A_{(N)}\| \leq \|G - G_{(N)}\| \to 0$$

where

$$A_{(N)}u = \int G_N(x,y)u(y)dy \qquad G_N = \sum_{|k| \leq N, |l| \leq N} \langle F, \varphi_{kl} \rangle \varphi_{kl}$$

Observe that G_N gives a finite rank operator $A_{(N)}$ since

$$A_{(N)}(u) = \int G_N(x,y)u(y)dy = \sum_{|l|,|k| \leq N} \int \varphi_{kl} e^{ikx} e^{ily} u(y) dy$$

so $A_{(N)}(u) \subset span_{|k| \leq N} \varphi_k$.

The operators $F \in L^2([-\pi,\pi]^2)$ that we defined are called Hilbert-Schmidt operators.

Example. The Wave Equation and Poisson Summation We go back to the wave equation. If $u_0, u_1 \in C^\infty(\mathbb{R})$, 2π-periodic then there exists a unique $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ such that $u_{tt} - u_{xx} = 0$ and $u(0,x) = u_0$, $u_t(0,x) = u_1$.

We can define a map depending on t with $(u_0, u_1) \mapsto (u(s,\cdot), u_t(s,\cdot))$. This defines a linear map

$$\sqcup_s : H^1([-\pi,\pi]) \times L^2([-\pi,\pi]) \to H^1([-\pi,\pi]) \times L^2([-\pi,\pi])$$

The mapping is continuous (follows from the fact the the problem can also be solved using Fourier Series). It is also adjoint(**Exercise**), just compute

$$\int_0^s \int_{-\pi}^{\pi} u_t(u_{tt} - u_{xx})$$

Alternatively we can choose $\varphi \in S(\mathbb{R})$ and consider the map

$$P : (u_0, u_1) \mapsto \left(\int_{\mathbb{R}} u(s, \cdot)\varphi(s)ds, \int_{\mathbb{R}} u_t(s, \cdot)\varphi(s)ds \right) = \int \sqcup_s \varphi(s)ds$$

Essentially, instead of fixing time, we are averaging over it. This is in fact compact on $H^1 \times L^2$, it is Hilbert-Schmidt and normal (we do not have time to prove this)

Now how his this related with poisson summation? We have two ways of computing P:

1. Solve the wave equation using the explicit formula

$$(u_0, u_1) \mapsto u = f(x + t) - g(x - t)$$

2. Fourier Series

We claim (but will not prove) the following very useful result

Theorem. *The eigenvalues of* P*,* $\lambda_j \in C$ *decrease to* 0 *and*

$$Tr(P) = \sum_{i=0}^{\infty} \lambda_i$$

converges. Furthermore the Poisson summation formula gives the relationship between $Tr(P)$ *computing with* #1 *and* #2*.*