1.1 Applications to Probability

$\mathcal{B}$ is the bernoulli space of outcome of coin toss. Where as usual $H \mapsto 1$, $T \mapsto 0$, and $\mathcal{B} \setminus B_{\text{deg}} \to (0,1]$. Let E be the set of outcomes in which an event occurs. Then the probability is $\mu_L(E)$

Define R_k, the Radamacher function

$$R_k(\omega) = 2a_k - 1, \qquad \omega = .a_1 a_2 \ldots$$

(sort of a heads you win, tails you lose function) The function looks like a square wave thingy.

We would like to measure how many heads in first N tosses. Let S_N be the number of heads in the first N tosses, then

$$S_N(\omega) = a_1 + \cdots + a_n, \qquad S_N : (0,1] \to \mathbb{N}$$

And we see that

$$S_N = \frac{1}{2}\left(\sum_{i=1}^{N} R_i(\omega) + N\right)$$

so

$$\boxed{S_N(\omega) - \frac{N}{2} = \frac{1}{2}\sum_{i=1}^{N} R_i(\omega)}$$

Let E be the number of heads in the first N tosses from $N/2$. The "distance" from $N/2$ that is.

$$E = \left\{\omega \in \mathcal{B} : \left|S_N(\omega) - \frac{N}{2}\right| > \epsilon\right\}$$

the question is what is $\lim_{N\to\infty} \mu_L(E_N)$

This is the same as saying that

$$E_N = \left\{\omega \in \mathcal{B}; \left|\frac{S_N}{N} - \frac{1}{2}\right|^2 > \epsilon^2\right\} = \left\{\omega; \frac{1}{4N^2}\left(\sum_{i=1}^{N} R_i\right)^2 > \epsilon^2\right\}$$

Before we proceed,

Theorem. ***Special Chebyshev*** *f is piecewise constant, non-negative on $[0,1]$ then*

$$\mu\{f(\omega) > \alpha\} \le \frac{1}{\alpha}\int_{[0,1]} f$$

Proof. If f is piecewise constant then we have intervals such that

$$\bigcup_i (x_i, x_{i+1}) = [0,1], \qquad (x_i, x_{i+1}) \text{ disjoint}$$

so $f(x) = c_i, x \in (x_i, x_{i+1}), c_i \geq 0$ and then

$$\int_{[0,1]} f = \sum_i c_i(x_{i+1} - x_i) \geq \sum_{c_i \geq \alpha} c_i(x_{i+1} - x_i) \geq \alpha \sum_{c_i > \alpha} (x_{i+1} - x_i) = \alpha\mu\{f(\omega) > \alpha\}$$

and so

$$\mu\{f(\omega) > \alpha\} \leq \frac{1}{\alpha} \int_{[0,1]} f$$

□

Theorem. ***Weak Law of Large Numbers***

$$\lim_{N\to\infty} \mu_L(E_N) = 0$$

Proof. E_N is equivalent to

$$\{\omega; f_N(\omega) > 4\epsilon^2 N^2\}, \qquad f_N(\omega) = \left(\sum_{i=1}^{N} R_i(\omega)\right)^2$$

then we apply Chebyshev with $\alpha = 4\epsilon^2 N^2, f_n = (\sum R_i)^2$ then

$$\int_{[0,1]} f = \int_{[0,1]} \left(\sum_{i=1}^{N} R_i^2 + \sum_{i=1}^{N} R_i R_j\right) = N$$

So

$$\mu(E_N) \leq \frac{N}{4\epsilon^2 N^2} = \frac{1}{4\epsilon^2 N}$$

this tends to 0 as $N \to \infty$ □

1.2 Continue Extending Measures

One of our methods was to define a measurable set, A, as one in which

$$\mu^*(C) = \mu^*(C \cap A) + \mu^*(C \cap A^c), \qquad \forall C \in 2^X$$

A second approach is to define a set $\mathcal{M}_F$ as follows

$$\mathcal{M}_F = \{B \in 2^x | \exists \text{ a sequence } A_i \in \mathcal{R}, \text{ such that } \mu^*(B \ominus A_i) = 0\}$$

then we define $\mathcal{M}$ as the set of countable unions of sets of M_F

On 2^X consider the function

$$d_\mu(A, B) = \mu^*(B \ominus A) \in [0, \infty]$$

Idea This is *almost* a metric on the power set. We see its, symmetric and non-negative, but $d(A, B) = 0$ does not mean that $A = B$, but thats alright. However, we have to check that it has the triangle inequality:

$$\mu^*(A \ominus B) \leq \mu^*(A \ominus C) + \mu^*(B \ominus C)$$

this follows from the fact that

$$(A \ominus B) \subset (A \ominus C) \cup (B \ominus C)$$

(check above relation for self)