

## 8 Extending Fourier Transform to $L^2(\mathbb{R})$ , Proving the Schwartz Functions are Dense in $L^2$

The Schwartz functions

$$S(\mathbb{R}) = \left\{ f : \mathbb{R} \rightarrow \mathbb{C}, f \in C^\infty, \sup_{x \in \mathbb{R}} \left| x^m \frac{d^l}{dx^l} f(x) \right| < \infty \forall m, l \right\}$$

Here are our main propositions for this section. These should always be in the back of our minds.

**Proposition.**  $S(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$

**Theorem.** The Fourier Transform  $\mathcal{F}$  extends by continuity to an isomorphism  $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ .

We remark that  $S(\mathbb{R})$  is a complete metric space (but not a Banach Space).  
Now we can define a norm on  $S$  by

$$\|f\|_k = \sum_{p+q \leq k} \sup_{x \in \mathbb{R}} \left| x^p \frac{d^q}{dx^q} f \right|$$

Each of the terms of the sum is a norm of  $S(\mathbb{R})$ . So  $S(\mathbb{R})$  with  $\|\cdot\|_k$  is a normed space (but not complete).

We define (not a norm) a metric

$$d(f, g) = \sum_{k=1}^{\infty} \frac{\|f - g\|_k}{1 + \|f - g\|_k} 2^{-k}, \quad \in [0, \infty)$$

This is well defined. To show its a metric we just have to show the triangle inequality.

**Lemma.** If  $\|\cdot\|$  is a norm, then

$$d(f, g) = \frac{\|f - g\|}{1 + \|f - g\|}$$

is a distance.

*Proof.* This is not very hard, its just algebra. □

**Exercise** Prove that  $S(\mathbb{R})$  is complete with respect to the metric defined above.

Now we already know

**Theorem.**  $\mathcal{F} : S(\mathbb{R}) \rightarrow S(\mathbb{R})$ .

**Definition.** A tempered distribution "generalized function" is a linear function  $u : S(\mathbb{R}) \rightarrow \mathbb{C}$

Now look at  $v \in L^2(\mathbb{R})$ , define

$$v(f) = \int_{\mathbb{R}} f(x)v(x)dx, \quad \frac{du}{dx}(f) = u \left( -\frac{df}{dx} \right)$$

Then  $L^2(\mathbb{R}) \hookrightarrow S'(\mathbb{R}) = \{u : S(\mathbb{R}) \rightarrow \mathbb{C}\}$ .  $S'(\mathbb{R})$  is the dual space of  $S(\mathbb{R})$ .

**Example.**  $\delta_0 : S(\mathbb{R}) \rightarrow \mathbb{C}$ . Then  $\delta_0(f) = f(0)$ . Because

$$H = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

And

$$H(f) = \int_{\mathbb{R}} f H d\mu = \int_y^{\infty} f(x)dx, \quad \forall f \in S(\mathbb{R})$$

Then in  $S'(\mathbb{R})$

$$\frac{d}{dx}H(f) = H \left( -\frac{df}{dx} \right) = - \int_0^{\infty} \frac{df}{dx} = f(0) = \delta_0(f)$$

To show that  $\mathcal{F}$  can be extended from a dense subset to a  $L^2$ . Now formally we want to show that if  $f_n \in S(\mathbb{R})$ ,  $f_n \rightarrow f$  in  $L^2$ . Then  $\mathcal{F}f_n = \hat{f}_n$  satisfies

$$\|\hat{f}_n - \hat{f}_m\|^2 = 2\pi\|f_n - f_m\|^2 \rightarrow 0, \quad n \rightarrow \infty$$

So then  $\hat{f}_n$  is Cauchy and thus convergent in  $L^2(\mathbb{R})$ . So we want to define

$$\mathcal{F}f = \lim_{n \rightarrow \infty} \hat{f}_n \quad f = \lim_{n \rightarrow \infty} f_n, \quad f_n \in S(\mathbb{R})$$

So that

$$\mathcal{F} : S(\mathbb{R}) \rightarrow S(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$$

## 8.1 Density of the Schwartz Functions

To prove this we need to show the existence of a sequence of functions  $f_n \in S_n$  for each  $f \in L^2(\mathbb{R})$  such that  $f_n \rightarrow f$ .

**Theorem. Uniqueness** If  $f'_n \rightarrow f$ ,  $f'_n \in S$ , then

$$\lim_{n \rightarrow \infty} \hat{f}'_n = \lim_{n \rightarrow \infty} f_n = \hat{f}$$

so that  $\mathcal{F}f$  is well defined.

*Proof.*  $f'_n \rightarrow f$ , then  $f'_n - f_n \rightarrow 0$  in  $L^2(\mathbb{R})$ . The Placherel implies  $\|\hat{f}'_n - \hat{f}\|^2 = (2\pi)\|f'_n - f\|^2 \rightarrow 0$   $\square$

The proof that  $S(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$ . Comes in 2 steps.

### First Step

**Theorem.** If  $f \in L^2(\mathbb{R})$  and

$$f_N(x) = \begin{cases} f(x), & |x| \leq N \\ 0, & |x| > N \end{cases}$$

then  $f_N \rightarrow f$  in  $L^2(\mathbb{R})$  as  $N \rightarrow \infty$

*Proof.*

$$\|f - f_N\|^2 = \begin{cases} 0, & |x| \leq N \\ -f(x), & |x| > N \end{cases}$$

$$\|f - f_N\|^2 = \int_{|x| \geq N} |f(x)|^2 dx \rightarrow 0, \quad \text{as } N \rightarrow \infty$$

because  $f$  is square integrable. □

**Second Step** Now it suffices to show that the functions above can be approximated by in  $L^2$  by sequences in  $S(\mathbb{R})$ .

We use Fourier Series. Suppose  $f$  vanishes outside radius of  $2R$ . Consider  $g(y) = f(2Ry)$ . We are sort of refitting the function so that  $g$  will vanish outside of  $[-1, 1]$ ,  $g \in L^2$ . Then by results in Fourier series,  $g = \sum c_n e^{iny}$  converges in  $L^2$ , i.e. the  $C^\infty$  function  $g_m(y) = \sum_{|n| \leq m} c_n e^{iny} \in C^\infty([-\pi, \pi])$  converges to  $g$  in  $L^2([-\pi, \pi])$ .

We claim the existence of a function  $\chi \in C^\infty(\mathbb{R})$  such that  $0 \leq \chi \leq 1$  and

$$\chi(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| \geq \pi \end{cases}$$

**Lemma.**  $g_m \rightarrow g$  in  $L^2([-\pi, \pi])$  then  $\chi g_m \rightarrow \chi g$  in  $L^2([-\pi, \pi])$ .

*Proof.*

$$\int_{-\pi}^{\pi} |\chi g_m - \chi g|^2 dx = \int_{-\pi}^{\pi} \chi^2 |g_m - g|^2 dx \leq \int_{-\pi}^{\pi} |g_m - g|^2 dx \rightarrow 0$$

□

**Lemma.** We claim that

$$\chi g_m \left( \frac{x}{2R} \right) \rightarrow f \quad \in L^2(\mathbb{R}) \quad \text{as } m \rightarrow \infty$$

and that

$$f_m = \chi g_m \left( \frac{x}{2R} \right) \in S(\mathbb{R})$$

*Proof.*  $f_m \in C^\infty(\mathbb{R})$  and  $f_m(x) \equiv 0$  with  $|x| \geq 2\pi R$ . So this is a smooth function with

$$\left| x^l \frac{d^q}{dx^q} f \right| \leq C_{n,l}, \quad C_{n,l} \text{ a constant}$$

so  $f_m \in S(\mathbb{R})$ , and its easy to see convergence. □

Now to finish the proof we need to construct  $\chi$ , the "cutoff" function.

**Theorem.** *If  $a < b$  in  $\mathbb{R}$  with  $\epsilon > 0$ ,  $\exists \chi : \mathbb{R} \rightarrow [0, 1]$  which is  $C^\infty$  such that*

$$\chi(x) = \begin{cases} 1, & x \in [a, b] \\ 0, & x \in (-\infty, a - \epsilon] \cup [b + \epsilon, \infty) \end{cases}$$

**Step 1** Observe that

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x < 0 \end{cases}$$

is in  $C^\infty$ . We prove this by induction. We want to show that

$$\frac{d^k}{dx^k} (e^{-\frac{1}{x}}) = \frac{p_k(x)}{x^{2k}} e^{-\frac{1}{x}}$$

where  $p_k$  is a polynomial, this is true because if we take a derivative

$$\frac{d^{k+1}}{dx^{k+1}} (e^{-\frac{1}{x}}) = \frac{x^2 p'_k - 2kx p_k + p_k}{x^{2k+2}} e^{-\frac{1}{x}}$$

Now we have

$$\lim_{x \rightarrow 0^+} \frac{d^k f}{dx^k} = \lim_{x \rightarrow 0^+} \frac{p_k(x)}{x^{2k}} e^{-\frac{1}{x}} \rightarrow 0$$

**Step 2** We can explicitly construct  $\chi$  now:

$$\chi = \frac{f(x - a + \epsilon) f(-x + b + \epsilon)}{f(-x + a + \epsilon) + f(x - a + \epsilon) + f(-x + b + \epsilon) + f(x - b + \epsilon)}$$

we see that this satisfies what we want. And we are done.

**Review Proof** We cut off the crap near infinity and get a sequence which converges to some function. So now it suffices to assume that  $f = 0$  in  $|x| \geq N$ . Now approximate in  $L^2$  using fourier series in  $L^1$ , which we can do since

$$L^2([-N, N]) \subset L^1([-N, N]), \quad \|f - \varphi_n\|_1 \leq (2N)^{1/2} \|f\|_2$$

(some stuff which I don't know what it means:)  $f \in S$

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx, \quad \sup_{\xi \in \mathbb{R}} |\hat{f}(\xi)| \leq \int_{\mathbb{R}} |f|$$

$\varphi_n \rightarrow f$  in  $L^1$  then

$$\sup_{\xi \in \mathbb{R}} |\hat{f} - \hat{\varphi}_n| \leq \|f - \varphi_n\|_1 \rightarrow 0$$