Practice for first test

I have decided to make the first test open-book, but only copies of Adams and Guillemin are permitted, no papers or other writings.

1. Set $X = [0,1] \times [0,1]$, the unit square in $\mathbb{R}^2$ and suppose that $f : X \longrightarrow [0,\infty)$ is Lebesgue measurable and such that

$$\mu_{\text{Leb}}\{x \in X; f(x) \geq n\} \leq 2^{-n} \ \forall \ n \in \mathbb{N}.$$

   Show that $\int_X f dx < \infty$.

2. Without using the equality of Riemann and Lebesgue integrals show that

$$\int_{\mathbb{R}} (1 + |x|)^{-1} dx = \infty$$

   and

$$\int_{\mathbb{R}} (1 + |x|^2)^{-1} dx < \infty.$$

3. Give, with proofs, an example of a non-negative Lesbesgue measurable function on $[0,1]$ which has finite Lebesgue integral but is not bounded.
4. Explain why the formula

$$\mu(A) = \int_A \exp(-x^2) dx$$

   defines a countably additive finite measure on the σ-ring, $\mathcal{M}$, of all Lebesgue measurable subsets of $\mathbb{R}$. Show that completion of $\mathcal{M}$ with respect to this measure is just $\mathcal{M}$ again.
5. Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose that $s_n : X \longrightarrow [0,\infty)$ is an increasing (i.e. pointwise non-decreasing) sequence of simple measurable functions with $\int_X s_n d\mu < 1$. Show that the set consisting of those points $x \in X$ where $s_n(x) \to \infty$ has μ-measure zero.