9 The Harmonic Oscillator

We know that $L^2(\mathbb{R})$ is Hilbert space and it is separable, so it MUST have a complete orthonormal basis. For $L^2([-\pi, \pi])$ we look that the 2π periodic solutions to the equation

$$\boxed{\frac{d^2}{dx^2} f = \lambda f}$$

and these generated such a basis. Now, for $L^2(\mathbb{R})$ we have the **Harmonic Oscillator**. This is the operator

$$H = \left(-\frac{d^2}{dx^2} + x^2\right)$$

we want to solutions to $Hu = \lambda u$:

$$\boxed{\left(-\frac{d^2}{dx^2} + x^2\right) u = \lambda u, \qquad u \in S(\mathbb{R})}$$

To do this, consider the two operators "Annihilation" and "Creation"

$$A = \left(\frac{d}{dx} + x\right), \qquad C = \left(-\frac{d}{dx} + x\right)$$

Now, a computation to establish at least one eigenfunction

$$\begin{aligned}\left(-\frac{d^2}{dx^2} + x^2\right) e^{-x^2/2} &= -\frac{d}{dx}(-xe^{-x^2/2}) + x^2 e^{-x^2/2} \\ &= -x^2 e^{-x^2/2} + e^{-x^2/2} + x^2 e^{-x^2/2}\end{aligned}$$

so $e^{-x^2/2}$ is an eigenfunction for $\lambda = 1$.

Now look at the operation of the operators A and C on each other

$$ACu = \left(\frac{d}{dx} + x\right)\left(-\frac{d}{dx} + x\right) u = -\frac{d^2 u}{dx^2} + x\frac{du}{dx} + u - x\frac{du}{dx} + x^2 u = \left(-\frac{d^2}{dx^2} + x^2 + 1\right) u$$

So we have the relation (when we include the computation for CAu)

$$\boxed{CAu = (H-1)u, \qquad ACu = (H+1)u}$$

so

$$[A, C]u \equiv (AC - CA)u = 2u$$

we already know that $He^{-x^2/2} = e^{-x^2/2}$, and $Ae^{-x^2/2} = 0$ and

$$HCu = (CA + 1)Cu = CAC + Cu = C(AC + 1)u = C(H + 2)u$$

So we now know that if $u \in S(\mathbb{R})$ then

$$HCu = C(H+2)u$$

Set $u = e^{-x^2/2}$ then we get the following relations by induction:

$$\begin{aligned} (H-1)e^{-x^2/2} &= 0 \\ (H-3)(Ce^{x^2/2}) &= 0 \\ &\vdots \\ H(C^j e^{-x^2/2}) &= (2j+1)C^j e^{-x^2/2} \end{aligned}$$

Proposition. *The functions*

$$\varphi_j = \frac{C^j e^{-x^2/2}}{\|C^j e^{-x^2/2}\|}$$

are a complete orthonormal basis of $L^2(\mathbb{R})$.

First $C^j e^{-x^2/2} \in L^2(\mathbb{R})$ because $C^j e^{-x^2/2} = p_j(x)e^{-x^2/2}$, p_j (unnormalized) **Hermite Polynomials**. And by induction

$$C^{j+1}e^{-x^2/2} = C(C^j e^{-x^2/2}) = C(p_j e^{-x^2/2}) = (xp_j - p_j')e^{-x^2/2} + (2xp_j + p_j')e^{-x^2/2}$$

So $p_j = 2^j x^j +$ lower order terms.

NB $p_0, \ldots, p_N$ span all polynomials of degree $\leq N$, because for a polynomial q_N,

$$q_N = \frac{c}{2^N} p_N + q_{N-1}$$

and so on (just pick first term then pick second...)

Now we need to show orthonormality, that is

$$\langle \varphi_j, \varphi_k \rangle = \int_{\mathbb{R}} \varphi_j \overline{\varphi_k} dx = \int_{\mathbb{R}} \varphi_j \varphi_k dx = 0, \qquad j \neq k$$

$[H - (2j+1)]\varphi_j = 0$, so we write $\varphi_j = \frac{1}{2j+1} H\varphi_j$, and the above becomes

$$\int_{\mathbb{R}} \left(\frac{H\varphi_j}{2j+1} \right) \varphi_k dx = \frac{1}{2j+1} \int_{\mathbb{R}} \left(-\frac{d^2}{dx^2}\varphi_j + x^2\varphi_j \right) \varphi_k dx$$

these are all Schwartz functions, so we cut off at infinity and integrate by parts,

$$\begin{aligned} &\lim_{N\to\infty} \int_{-N}^{N} -\frac{d}{dx}(\psi)\varphi_k dx, \qquad \psi = \frac{d}{dx}\varphi_j \\ &= \lim_{N\to\infty} \int_{-N}^{N} \psi \frac{d}{dx}\varphi_k - \psi\varphi \Big|_{-\infty}^{\infty} \end{aligned}$$

(basically, this shifts differentation to the left)

then we get

$$\frac{1}{2j+1}\int_{\mathbb{R}} \varphi_j \left(-\frac{d^2}{dx^2}+x^2\right)\varphi_k dx = \frac{1}{2j+1}\int_{\mathbb{R}} \varphi_j H\varphi_k dx$$

But $H\varphi_k = (2k+1)\varphi_k$ so the above is just

$$\frac{2k+1}{2j+1}\int_{\mathbb{R}} \varphi_j\varphi_k dx$$

then

$$\langle \varphi_n, \varphi_k\rangle = \frac{2k+1}{2j+1}\langle \varphi_j, \varphi_k\rangle \Longrightarrow \langle \varphi_j, \varphi_k\rangle = 0$$

Now we have to prove **completeness**. i.e. in $u \in L^2(\mathbb{R})$ and $\int_{\mathbb{R}} u\varphi_j = 0, \forall j$, then $u = 0$. In fact if we have shown this then

$$\int_{\mathbb{R}} u\left(\sum_{j=0}^{N} c_j\varphi_j\right) dx = 0 \qquad \text{and} \qquad \int_{\mathbb{R}} up_N e^{-x^2/2}dx = 0$$

for all polynomials, because we know that combinations of the φ_j are all polynomials. Can you show that this implies $u = 0$ (a.e. in $L^2(\mathbb{R})$)? Suppose that u has compact support (so its non-zero in a bounded set) then

$$\int_{-N}^{N} (ue^{-x^2/2})p(x)dx = 0, \qquad \forall\text{polynomials}$$

But the polynomials are dense in L^2. (Think about it, this implies completeness of our problem)

So any $f \in L^2(\mathbb{R})$ can be written as the sum of its "Fourier Series"

$$f = \sum_{i=0}^{\infty} c_j\varphi_j, \qquad c_j = \int_{\mathbb{R}} f\varphi_j dx \qquad \varphi_j = \frac{C^j e^{-x^2/2}}{\|C^j e^{-x^2/2}\|}$$

There are many consequences of this. For example if $f \in L^2(\mathbb{R})$ the solution to the equation $Hu = f$ is

$$u = \sum_{i=0}^{\infty} \frac{c_j(f)}{2j+1}\varphi_j, \qquad c_j(f) = \int_{\mathbb{R}} f\varphi_j dx$$

In some weak sense $Hu = f$. Its "weak" because we don't really k now about differentiation. So our weak formulation is really that

$$\left(-\frac{d^2}{dx^2}+x^2\right)u = f \qquad u, f \in L^2(\mathbb{R})$$

$$\int u\left(-\frac{d^2}{dx^2}\varphi + x^2\varphi\right)dx = \int f\varphi, \qquad \forall\varphi \in S(\mathbb{R})$$

Completeness of solutions of the harmonic oscillator. $\exists \varphi_0, \varphi_1, \varphi_2, \ldots$ in $S(\mathbb{R})$ with φ_0 the ground state, which means that $H\varphi_0 = \varphi_0$, so $\varphi_0 = \frac{1}{2\pi}\exp(-x^2/2)$, $\|\varphi_0\|_2 = 1$, φ_j we define as

$$\varphi_j = \frac{C^j \varphi_0}{\|C^j \varphi_0\|_L}, \qquad H\varphi_j = (2j+1)\varphi_j$$

and

$$span\{\varphi_0, \ldots, \varphi_n\} = \{p(x)e^{-x^2/2} | p\ poly\ deg(p) \leq N\}$$