Proposition. *μ_L is countably additive.*

Proof. Given $A_i \in \mathcal{R}$, $A_i \cap A_j = \emptyset$, we know that $\bigcup_{i=1}^{\infty} A_i = A \in \mathcal{R}_L$ we must show that $\mu_L(A) \leq \sum_{i=1}^{\infty} \mu_L(A_i)$, and then we are done, since we already know by the previous theorem that $\mu_L(A) \geq \sum_{i=1}^{\infty} \mu_L(A_i)$.

Lemma. *If $A \in \mathcal{R}_L$, $\mu(A) > 0$*

1. *$\exists F \in \mathcal{R}_L$ such that F is closed and $F \subset A$ and $\mu_L(F) \geq \mu_L(A) - \epsilon$*
2. *If $A \in \mathcal{R}_L$ then $\exists G \in \mathcal{R}_L$, G open and $G \supset A$ and $\mu(G) \leq \mu(A) + \epsilon$.*

Proof. It suffices to show this for multi-intervals. Let

$$I = (a_1, b_1) \times \cdots \times (a_n, b_n)$$

(assume $b_i > a_i$), I have denoted the interval here as open, but it may be open closed or semi-open. Define F as follows

$$F = [a_1 + \delta, b_1 - \delta] \times \cdots \times [a_n + \delta, b_n - \delta] \subset I$$

then

$$\mu(F) = \prod_{j=1}^{n} (b_j - a_j - 2\delta) = \prod_{j=1}^{n} (b_j - a_j) - \delta F(\delta) < \mu_L(A).$$

if we choose δ small enough ($\delta F(\delta)$ is a polynomial in δ and vanishes as $\delta \to 0$.

We can define G similarly by

$$G = (a_1 - \delta, b_1 + \delta) \times \cdots \times (a_n - \delta, b_n + \delta) \supset I$$

We continue with our proof.

Given $\delta > 0$ apply the above lemma to get $F \subset A$, $\mu_L(F) \geq \mu(A) - \delta$, given $\epsilon > 0$ apply lemma to each A_i to get G_i open such that $A_i \subset G_i$ and $\mu(G_i) \leq \mu(A_i) + \epsilon/2^i$. Now $F \subset A = \bigcup^\infty A_i$ and so F is closed and bounded and thus compact, by Heine-Borel. So there is a finite subcover. This implies that

$$F \subset \bigcup_{i=1}^N A_i$$

for some N(since we have a finite subcover, we might as well just take the first N). But we know that we have finite additivity of μ, so

$$\mu(F) \leq \sum_{i=1}^N \mu(G_i)$$

and

$$\mu(A) - \delta \leq \mu(F) \leq \sum_{i=1}^N \mu(G_i) \leq \sum_{i=1}^N \left(\mu(A_i) + \frac{\epsilon}{2^i}\right) \leq \sum_{i=1}^\infty \mu(A_i) + \epsilon$$

then we have

$$\mu(A) - \delta \leq \sum_{i=1}^\infty \mu(A_i) + \epsilon$$

so $\mu_L(A) \leq \sum_{i=1}^\infty \mu_L(A_i)$ □

We try to enlarge $\mathcal{R}$. First we try to measure every subset of X. Suppose $B \subset X$. Try to cover B by a countable collection of elements of $\mathcal{R}$.

Is $B \subset \bigcup_{i=1}^\infty A_i$, $A_i \in \mathcal{R}$

Definition. Outer Measure: $\mu^* : 2^X \to [0, \infty]$ has two conditions

1. If it is not possible to cover the set then we define $\mu^*(B) = \infty$
2. If it is possible we say

$$\mu^* = \inf_{\text{covers}} \sum_{i=1}^\infty \mu(A_i) \in [0, \infty]$$

Lemma. *If $A \in \mathcal{R}$ then $\mu^*(A) = \mu(A)$*

Proof. A is a cover of itself, so automatically, $\mu^*(A) \leq \mu(A)$. Suppose $\{A_i\}_{i=1}^\infty$ is a cover of A, and $A_i \in \mathcal{R}$. We would like that

$$\mu(A) \leq \sum_{i=1}^\infty \mu(A_i)$$

Because then, in particular $\mu(A) \leq \inf \sum \mu(A_i) = \mu^*(A)$ and we would be done.

Set $A_i' = A_i \cap A \in \mathcal{R}$, then $A = \bigcup_{i=1}^{\infty} A_i'$. So the A_i' are in $\mathcal{R}$, but they are not disjoint. So to make them disjoint we do the following

$$\begin{aligned} A_1'' &= A_1' \\ A_2'' &= A_2' \setminus A_1' \\ &\vdots \\ A_N'' &= A_N' \setminus \left(\bigcup_{i=1}^{N-1} A_i' \right) \subset \mathcal{R} \end{aligned}$$

Then $A_N' \subset \bigcup_{i=1}^{N} A_i''$ and this construction means that $A_k' \cap A_l' = \emptyset$, $h \neq l$ and now

$$A = \bigcup_{i=1}^{\infty} A_i'', A_j'' \cap A_k'' = \emptyset \Rightarrow \mu(A) = \sum_{i=1}^{\infty} \mu(A_i'') \leq \sum_{i=1}^{\infty} \mu(A_i')$$

the last part follows since $A_i'' \subset A_i'$ and since $A_i' \subset A_i$ then

$$\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)$$

and we are done. □

Caratheodory's Idea What subsets of X should be measurable.

Definition. Measureable Set Call $\mathcal{M}$ the set of measurable sets. Then

$$(\dagger) \qquad A \in \mathcal{M} \Leftrightarrow \mu^*(C) = \mu^*(C \cap A) + \mu^*(C \cap A^c), \forall C \in 2^X$$

Definition. Set of Measure 0 Consider μ countably additive, $B \subset X$ has measure 0 if $\mu^*(B) = 0$, i.e. $\forall \epsilon > 0\ \exists A_i \subset \mathcal{R}$ such that $B \subset \bigcup_{i=1}^{\infty} A_i$ such that

$$\sum_{i=1}^{\infty} \mu(A) < \epsilon$$

Theorem. *A set of measure* 0 *has the Caratheodory property.*

Proof. Assume $\mu^*(A) = 0$. Then $\mu^*(C \cap A) = 0 \forall C$, because a cover of A is a cover of $C \cap A$. So then we try to show $\mu^*(C) = \mu^*(C \cap A^c)$.

It is clear that $\mu^*(C) \geq \mu^*(C \cap A^c)$. And by subadditivity $\mu^*(C \cap A) + \mu^*(C \cap A^c) \geq \mu^*(C)$, so $(\dagger)$ holds.

□