

## 13.2 Operators

Remember Riesz representation theorem in Hilbert Space.

**Theorem.** If  $B : H \rightarrow H$ ,  $H$  a Hilbert space,  $B$  a linear map then  $B(c_1f_1 + c_2f_2) = c_1Bf_1 + c_2Bf_2$ , then  $B$  is continuous is equivalent to saying  $B$  is bounded, i.e.  $\exists C$  such that  $\|Bf\|_H \leq C\|f\|_H, \forall f \in H$ .

*Proof.* Suppose  $B$  is continuous and linear then  $B$  is continuous at 0, and  $B(0) = 0$ , continuity implies that  $B^{-1}(\{\|g\| < \epsilon\}) \supset \{\|f\| < \delta\}$  for some  $\delta > 0$ . That is  $\|f\|_H \leq \delta/2$  implies  $f \in \{\|f\| < \delta\}$  implies that  $Bf \in \{\|g\| < 1\}$  so  $\|Bf\| < 1$ , so  $\|f\|_H \leq \delta/2 \Rightarrow \|Bf\| \leq 1$ . But this is boundedness. For general  $f \in H, f \neq 0$

$$B\left(\frac{f\delta/2}{\|f\|}\right) \leq \frac{\delta}{2} = \frac{\delta/2}{\|f\|}Bf \quad \left\|B\left(\frac{\delta/2f}{\|f\|}\right)\right\| \leq 1$$

so

$$\|Bf\| \leq \frac{2}{\delta}\|f\|$$

Conversely, if  $\|Bf\| \leq C\|f\|, \forall f \in H$ , then  $\|B(f - g_j)\| \leq C\|f - g_j\|$  and  $\|Bf - Bg_j\| \leq C\|f - g_j\|$  so if  $g_j \rightarrow f$  then  $\|f - g_j\| \rightarrow 0 \Rightarrow \|Bf - Bg_j\| \rightarrow 0 \Rightarrow Bg_j \rightarrow Bf$ . So  $B$  is continuous.  $\square$

**Example.** Before we showed that  $f_0, f_1 \in C^\infty(\mathbb{R})$  is of period  $2\pi$ , then  $\exists$  unique solution  $u(t, x)$  to the wave equation with  $u(0, x) = f_0$  and  $\partial_t u(0, x) = f_1$ . Claim: if we define

$$\begin{aligned} \square_T : C_\pi^\infty(\mathbb{R}) \times C_\pi^\infty(\mathbb{R}) &\rightarrow C_\pi^\infty(\mathbb{R}) \times C_\pi^\infty(\mathbb{R}) \\ (f_0, f_1) &\mapsto (u(T, \cdot), \partial_T u(T, \cdot)) \end{aligned}$$

This is linear. Either directly or from Fourier, this map is "bounded"

$$\|\partial_t u(T, \cdot)\|^2 + \|u(T, \cdot)\|^2 + \|\partial_x u(T, \cdot)\|^2 \leq C(\|\partial_x f_0\|^2 + \|f_0\|^2 + \|f_1\|^2)$$

So now we know if a linear map  $B : H \rightarrow H$  is continuous if and only if its bounded ( $\|Bu\| \leq C\|u\|, \forall u \in H$ ).

**Definition. Operator Norm**

$$\|B\| = \sup\{\|Bu\|; u \in H, \|u\| = 1\}$$

so  $\|Bu\| \leq \|B\|\|u\|, \forall u \in H$

Let  $\mathcal{B}(H)$  be the space of all bounded operators on  $H$ . Then  $\mathcal{B}(H)$  is a Banach space with norm  $\|B\|$ . It is clear that  $\|-\|$  is a norm, because  $\|(B_1 + B_2)u\|_H = \|B_1u + B_2u\|_H$ . By the triangle inequality on  $H$ ,  $\leq \|B_1u\| + \|B_2u\| \leq (\|B_1\| + \|B_2\|)\|u\|$ .

Note also that that  $\mathcal{B}(H)$  is an algebra under operator composition  $H \xrightarrow{B_1} H \xrightarrow{B_2} H$ . And  $B_2 \circ B_1 \in \mathcal{B}(H)$ , because  $\|B_2B_1u\|_H \leq \|B_2\|\|B_1u\| \leq \|B_2\|\|B_1\|\|u\|$ .

This is called **the strong property of bounded operators**:  $\|B_2B_1\| \leq \|B_2\|\|B_1\|$ .

Now we talk about adjoint operators.

**Definition. Adjoint Operation.** If  $B \in \mathcal{B}(H)$ ,  $\exists B^* \in \mathcal{B}(H)$  such that

$$\langle Bu, v \rangle_H = \langle u, B^*v \rangle_H \quad \forall u, v \in H$$

Now we have to define explicitly what  $B^*$  is.

**Definition. The Adjoint** Given  $v \in H$  define  $B^*v = w \in H$ , we want  $\langle u, w \rangle = \langle Bu, v \rangle \forall u \in H$ .

Given  $v \in H$

$$H \ni u \mapsto \langle Bu, v \rangle \in \mathbb{C}$$

This is a continuous linear map (functional).

But remember the Riesz Representation Theorem: If  $F : H \rightarrow \mathbb{C}$  is continuous and linear,  $\exists w \in H$  such that  $F(u) = \langle u, w \rangle, \forall u \in H$ . So  $\langle Bu, v \rangle = \langle u, w \rangle$  for unique  $w$  by Riesz. So this is how we define  $B^*v = w$ .

Check

### 1. Linearity

$$\langle Bu, c_1v_1 + c_2v_2 \rangle = \bar{c}_1\langle Bu, v_1 \rangle + \bar{c}_2\langle Bu, v_2 \rangle = \langle u, B^*(c_1v_1 + c_2v_2) \rangle = \langle u, c_1B^*v_1 + c_2B^*v_2 \rangle$$

.

### 2. Bounded

$$|\langle Bu, v \rangle| = |\langle u, w \rangle| \leq \|B\|\|u\|\|v\|, \quad \forall u$$

Set  $u = w$ , then  $\|w\|^2 \leq \|B\|\|w\|\|v\|$  which implies  $\|w\| = \|B^*v\| \leq \|B\|\|v\|$  so  $\|B^*\| \leq \|B\|$ .

This is sort of a generalized integration by parts.

**Question.** When does  $B \in \mathcal{B}(H)$  map  $\{u \in H; \|u\| \leq 1\}$  into a compact set? Such operators are said to be **compact operators**.

**Example.** Suppose  $B$  has finite rank (i.e.  $\mathcal{B}(H) = V \subset H$  is finite dimensional). Choose an orthonormal basis  $\underbrace{\varphi_1, \dots, \varphi_N}_{\text{span } V}, \varphi_{N+1}, \dots$  for  $B(H)$ . So  $f \in B(\{\|u\| \leq 1\})$  have equismall tails (in fact no tails at all)

**Theorem.** An element  $B \in \mathcal{B}(H)$  is compact if and only if  $\exists S_N$  a sequence of finite rank operators such that  $S_N \rightarrow B$  in  $\mathcal{B}(H)$  ( $\|S_N - B\| \rightarrow 0, N \rightarrow \infty$ ). i.e. the space of compact operators is the closure in  $\mathcal{B}(H)$  of the finite rank operators.

**Definition.** Call  $\mathcal{C}(H)$  the set of all compact operators.

**Corollary.**  $\mathcal{C}(H)$  is a closed  $*$ -subalgebra of  $\mathcal{B}(H)$  and a two sided ideal. That is we have to show that  $B$  compact implies  $B^*$  compact. To show its a two sided ideal we must show  $B_1$  or  $B_2$  compact then  $B_1 B_2$  compact, and  $B_1$  compact and  $B_2$  bounded implies that  $B_1 B_2$  compact.

*Proof.*

$$B_1 B_2(\{\|u\| \leq C\}) \subset B_1(B_2(\{\|u\| \leq C\})) \subset B_1(\{v \in H; \|v\| \leq \|B\|c\}) \subset \text{compact set}$$

□

Claim that  $S$  of finite rank then  $S^*$  has finite rank. Also if  $B_1 B_2 \in \mathcal{B}(H)$ , then  $(B_1 B_2)^* = B_2^* B_1^*$ . Now  $B_2 B_1 = (B_1^* B_2^*)^* \Rightarrow 2$  sided ideal.

Compact operators behave like matrices. In fact finite rank operators are essentially matrices.

Now some more theorems on bounded operators.

**Example.** Suppose  $\{\varphi_i\}_{i=1}^\infty$  is a complete orthonormal basis, then by Bessel,

$$f = \sum_{i=1}^{\infty} \langle f, \varphi_i \rangle \varphi_i$$

converges.

Now consider the following function, which will be important later

$$\pi_{(N)} = \sum_{i \geq N} \langle f, \varphi_i \rangle \varphi_i$$

This is a bounded operator, because

$$\|\pi_{(N)} f\|^2 = \sum_{i \geq N} |\langle f, \varphi_i \rangle|^2 \leq \|f\|^2$$

and  $\pi_{(N)} \varphi_N = \varphi_N$ , and  $\|\pi_{(N)}\| = 1$  and it is not too hard to see that  $\pi_{(N)}^2 = \pi_{(N)}$ , which makes  $\pi_{(N)}$  **idempotent** or a **projection** and  $\pi_{(N)}^* = \pi_{(N)}$ .

Can restate compactness of  $S \subset H$  as  $S$  closed, bounded and given  $\epsilon > 0$ ,  $\exists N$  such that  $\|\pi_{(N)} f\| < \epsilon$ .