4 Some Functional Analysis

We define $\mathcal{L}(\mu, \mathcal{F}) = \{f : X \to \mathbb{R} | \text{measurable and } \int_E |f| d\mu < \infty\}$. Then we define $L^1(X, \mu) = \{x \in \mathcal{L}(\mu, \mathcal{F})\}/a.e.$. $f \in L^1(X, \mu)$ is an equivalence class of elements of $\mathcal{L}(\mu, \mathcal{F})$, that is $f = [g] = \{g' \in \mathcal{L}(\mu, \mathcal{F}) | g - g' = 0 \text{ off a set of measure } 0\}$. We know $\int_E |f| d\mu = 0 \Rightarrow |f| = 0$ except on a set of measure 0. We want to show that the equivalence class defined above is in fact an equivalence class. We check that $f - g = 0$ on $X \setminus E$ for some set E, $\mu(E) = 0$ is an equivalence relation. We check transitivity $f - g = 0$ on $X \setminus E$, $\mu(E) = 0$ and $g - h = 0$ on $X \setminus F$, $\mu(F) = 0$, then $f - h = 0$ on $X \setminus (E \cup F)$, $\mu(E \cup F) = 0$. So in fact $L^1(X, \mu) = \{[f]_{a.e.} | f \in \mathcal{L}(\mu, \mathcal{F})\}$.

Theorem. *$L^1(X, \mu)$ is a metric space with the metric $d(f, g) = \int_E |f - g| d\mu$.*

Proof.

1. $d : L' \times L' \to [0, \infty)$, $d([f], [g]) = \int_E |f - g| d\mu$, $f \in [f], g \in [g]$. It is clear that working with equivalence classes doesn't matter so its well defined (also $d(f, g) = 0 \Rightarrow [f] = [g]$)

2. $d(f, g) = d(g, f)$, clear from definitions.

3. $d(f, g) \leq d(f, h) + d(h, g)$, $\forall f, g, h \in L^1(X, \mu)$. We write $\|f\|_{L^1} = \int_X |f| d\mu$. So we would like to show that $\|f - g\|_{L^1} \leq \|f - h\|_{L^1} + \|h - g\|_{L^1}$. Set $f' = f - h, g' = h - g$ so $\|f' + g'\|_{L^1} \leq \|f\|_{L^1} + \|g'\|_{L^1}$. This is true by monotonicity and linearity of the integral.

□

Definition. Normed Space. Conditions on a norm

1. $L^1(X, \mu)$ is a vector space (over $\mathbb{R}$),
2. $\| - \| : L^1(X, \mu) \to [0, \infty)$ satisfying
    (a) $\|f\| = 0 \Leftrightarrow f = 0$
    (b) $\|cf\| = |c| \|f\|$
    (c) $\|f + g\| \leq \|f\| + \|g\|$

We go ahead and do the same sort of construction but for complex functions. We have $(X, \mathcal{F}, \mu)$. Now $f : X \to \mathbb{C}$. We have the following definitions

1. $f : X \to \mathbb{C}$ is measurable $\Leftrightarrow$ $f = u + iv$, u, v are measurable.
2. f integrable iff $|f| = (|u|^2 + |v|^2)^{1/2}$ is integrable over E, i.e. $\int_E |f| d\mu < \infty$.

If f is integrable and complex valued we say $f \in \mathcal{L}(X, \mu)$. Its a linear space for all of the same reasons as $\mathcal{L}^1$.

Theorem. *$f \in \mathcal{L} \Rightarrow \overline{f} \in \mathcal{L}$ and $\int_E \overline{f} d\mu = \overline{\int_E f d\mu}$, and $|\int_E f d\mu| \leq \int_E |f| d\mu$.*

Proof. Let $a = \int_E f d\mu \in \mathbb{C}$, then

$$|a| = \frac{\overline{a}}{|a|} \int_E f d\mu = \int_E \frac{\overline{a}}{|a|} f d\mu$$

Write this in the form $\int u d\mu + i \int v d\mu$, but $i \int v d\mu = 0$ (because its an absolute value), so

$$\int_E u d\mu \leq \int_E |u| d\mu \leq \int_E |g| d\mu = \int_E |f| d\mu$$

□

We defined the normed space $L^1(X, \mu)$. This is also a metric with metric $d(f, g) = \|f - g\|$. We can ask questions like is this complete?

Definition. A **Banach Space** is a complete normed space. i.e. every Cauchy sequence in $L^1(X, \mu)$ is convergent.

Theorem. *$L^1(X, \mu)$ is a Banach Space.*

Proof. A Cauchy sequence is a sequence with its limit point stolen, so we want to construct a limit for the sequence $f_j \in L^1(X, \mu)$ to show that it exists. For f_j and $\forall \epsilon$, $\exists N$ such that $n, m \geq N$ then $\|f_n - f_m\| < \epsilon$.

Lemma. *If $\exists$ a convergent subsequence of a Cauchy sequence then the Cauchy sequence converges.*

Proof. trivial □

First a corollary of the Lebesgue dominated convergence theorem.

Corollary. *If $\sum_{n=1}^{\infty} \int_E |f_n| d\mu < \infty$ then $\sum_{n=1}^{\infty} f_n$ converges absolutely a.e. on E.*

Proof. Let $g = \sum_{n=1}^{\infty} |f_n|$ Then

$$\int_E g d\mu = \sum_{n=1}^{\infty} \int_E |f_n| d\mu < \infty$$

so $g \in L^1$ and g is finite a.e. on E. So

$$\sum_{n=1}^{\infty} f_n$$

converges absolutely on E. □

Suppose we have a sequence f_i such that $\|f_n - f_m\| < \epsilon$ for $n, m \geq N$. Choose a subsequence f_k such that $\|f_{n+1} - f_n\| \leq 2^{-n}$. Set $g_1 = f_1$, $g_2 = f_2 - f_1$ and in general $g_n = f_n - f_{n-1}$ and $f_n = \sum_{k=1}^{n} g_k(x)$. Now, $\sum_{k=1}^{n} \|g_k\|_1$ converges, since

$$\sum_{k=1}^{n} \|g_k\|_1 \leq \|g_1\|_1 + \sum_{n=2}^{n} 2^{-k} \leq \|f_1\| + \frac{1}{2}, \qquad \forall n$$

Now define

$$g(x) = \sum_{i=1}^{\infty} |g_k(x)|$$

this exists as an extension of the real numbers and most importantly

$$\int_X |g| d\mu = \int_X \sum_{k=1}^{\infty} |g_k(x)| d\mu = \sum_{k=1}^{\infty} \int_X |g_k| d\mu = \sum \|g_k\| < \infty$$

so $g \in L^1$. So in fact the sum converges pointwise and off a set of measure 0. Define a new function $\tilde{g}$ which is just g, but define it as 0 where g does not exist. Then obviously $\tilde{g} \in L^1$.

Now the series

$$\sum_{k=1}^{N} g_k(x) = f_N$$

converges **pointwise** and **absolutely**. And furthermore, $|\sum_{k=1}^{N} g_k(x)| \leq \tilde{g}$. Suppose the series converges to f. By the lebesgue dominated convergence theorem

$$\lim_{n\to\infty} \int |f_n - f| d\mu = \int \lim_{n\to\infty} |f_n - f| \to 0$$

so $\|f_n - f\| \to 0$ as $n \to \infty$ and we are done.

$\square$