Test 2 with solutions

This test is closed book. You are not permitted to bring any books, notes or such material with you. You may use theorems, lemmas and propositions from the book or from class.

Note that most of the solutions are relatively short - this is likely to be the case in the final as well!

1. If $f_n \in L^2([0,1])$ is a convergent sequence, with respect to the L^2 norm show that there is a subsequence which converges pointwise almost everywhere.

   Solution:- Since $[0,1]$ has finite measure we know that $L^2([0,1]) \hookrightarrow L^1([0,1])$ and in fact

   $$\|f\|_{L^1} \leq \|f\|_{L^2} \ \forall \ f \in L^2([0,1]).$$

   Now, from results in class we know that any sequence which converges in $L^1([0,1])$ has a subsequence which converges almost everywhere. It follows form the estimate above that if $\{f_n\}$ converges in L^2 it converges in L^1 and hence has a subsequence which converges almost everywhere in $[0,1]$.

2. Suppose that $f \in L^2([-\pi,\pi])$ has Fourier coefficients c_k satisfying

   $$\sum_{k\in\mathbb{Z}} k^2|c_k|^2 < \infty.$$

   Show that f is continuous on $[-\pi,\pi]$.

   Solution:- Since $f \in L^2([0,1])$ we know that its Fourier series converges to f in $L^2([-\pi,\pi])$

   $$f(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} c_k e^{ikx}.$$

   Now, consider the Fourier series itself, under the assumption (1). If $n \geq m > 0$ then

   $$|\sum_{m\leq|k|\leq n} c_k e^{ikx}| \leq \sum_{m\leq|k|\leq n} |c_k| \leq (\sum_{m\leq|k|\leq n} k^2|c_k|^2)^{\frac{1}{2}}(\sum_{m\leq|k|\leq n} \frac{1}{k^2})^{\frac{1}{2}}$$

   where we have used Schwarz' inequality. From the assumed convergence it follows that the right hand side is arbitrarily small if m is large enough. That is, the Fourier series itself is Cauchy in the uniform norm, hence uniformly convergent. It follows that the limit of this series is continuous and that it is a representative of f (this is what continuity of f means, it has a continuous representative).

3. Let f_i, $i \in \{0, 1, 2, 3, \dots\}$ be the sequence obtained by orthonormalization (the Gramm-Schmid process) of $x^k e^{-\frac{x^2}{2}}$. Show that for each n, the Fourier transform $\widehat{f_n}$ is linearly dependent on $f_0, \dots, f_n$.

Solution:- The Gram-Schmid process replaces the $g_k = x^k e^{-\frac{x^2}{2}}$ by f_i where each f_i is a linear combination of the g_k for $k \leq i$. Thus it is enough to show that g_n itself is linearly dependent on the Fourier transforms $\widehat{f_k}$, $k = 0, \dots, n$. Now, we also know that $\widehat{f_0} = cf_0$ for some non-zero constant c, so the statement is true for $n = 0$. We can proceed by induction, assuming that we have already shown that the statement is true for $n \leq p$ and then just prove it for $p + 1$. In fact the Fourier transform satisfies

$$\widehat{t^k f(t)}(\tau) = (-i)^k \frac{d^k}{d\tau^k} \widehat{f}(\tau).$$

Again by induction we therefore know (in fact we showed in class) that

$$\widehat{t^k f_0(t)}(\tau) = (c_k \tau^k + p(\tau)) f_0(\tau),$$

where p is a polynomial of degree at most $k - 1$ and $c_k \neq 0$. Thus by induction we see that the Fourier transform of g_{p+1} is a linear combination of the g_k for $0 \leq k \leq p + 1$.

4. Show that if $f : [-\pi, \pi] \longrightarrow \mathbb{C}$ is a bounded measurable function which satisfies $\int_{[-\pi,\pi]} x^{2k} f(x) dx = 0$ for all non-negative integers $k = 0, 1, 2, \dots$ then there is an odd function $g(x) = -g(-x)$, such that $f(x) = g(x)$ for almost all $x \in [-\pi, \pi]$.

Solved:- Since f is bounded and measurable, it is in $L^2([-\pi, \pi])$. Consider the function $h(x) = \frac{1}{2}(f(x) + f(-x))$. This is even and in $L^2([-\pi, \pi])$. Moreover it satisfies

$$\int_{[-\pi,\pi]} x^k h(x) dx = 0 \; \forall \; k = 0, 1, \dots.$$

For even powers this follows from the assumption on f since

$$\int_{[-\pi,\pi]} x^{2k} f(-x)dx = \int_{[-\pi,\pi]} x^{2k} f(x)dx.$$

On the other hand for odd powers

$$\int_{[-\pi,\pi]} x^{2k+1} f(-x)dx = -\int_{[-\pi,\pi]} x^{2k+1} f(x)dx$$
$$\Longrightarrow \int_{[-\pi,\pi]} x^{2k+1} h(x)dx = 0.$$

Now, we know that polynomials are dense in $L^2([-\pi,\pi])$ so we can choose a sequence $p_n(x) \to \overline{h(x)}$ in $L^2([-\pi,\pi])$. Thus it follows that $p_n(x)h(x) \to |h(x)|^2$ in $L^1([-\pi,\pi])$ and hence that

$$\int_{[-\pi,\pi]} |h(x)|^2 dx = 0 \Longrightarrow h = 0 \text{ almost everywhere.}$$

That $h = 0$ almost everywhere implies that

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = h(x) + g(x)$$

where g is an odd function.