

6 Completeness of Fourier Series

Our main theorem:

Theorem. If $f \in L^1([-\pi, \pi])$ and

$$a_n = \frac{1}{2\pi} \int_{[-\pi, \pi]} f(x) e^{-inx} dx = 0, \quad \forall n$$

then $f = 0$ a.e.

6.1 Proof 1

Steps

1. Replace f by

$$g(x) = \int_{-\pi}^x f(t) dt - C$$

where C is chosen so that $\int_{[-\pi, \pi]} g dx = 0$. At this step we would like to prove that **1)** The Fourier coefficients of g also vanish. **2)** g is continuous. We have already proved the second one somewhere.

2. Replace g by h :

$$g \longrightarrow h = \int_{-\pi}^x g(t) dt - C'$$

h is continuously differentiable and all fourier coefficients vanish.

OUR MAIN STEP:

Theorem. If f is continuous on $[-\pi, \pi]$ and differentiable at $x_0 \in (-\pi, \pi)$, then the Fourier series of f

$$\sum_{n \in \mathbb{Z}} c_n \varphi_n(x_0) = f(x_0)$$

(i.e. the series converges to $f(x_0)$)

(**Note:** We are going to apply this to h , we conclude that $h = 0$, g is the derivative of h so $g = 0$. And then the final step is to show $g = 0 \Rightarrow f = 0$ a.e.)

Proof. We want to show that $S_n(f)(x_0) \rightarrow f(x_0)$. Now

$$S_n(f)(x_0) = \sum_{k=-n}^n c_k \varphi_k(x_0) = \frac{1}{2\pi} \sum_{k=-n}^n \int_{-\pi}^{\pi} f(x) e^{-ikx+ikx_0} dx \Rightarrow$$

$$S_n(f)(x_0) = \int_{-\pi}^{\pi} f(x) D_n(x_0 - x) dx, \quad D_n(s) = \frac{1}{2\pi} \sum_{k=-n}^n e^{iks}$$

so now

$$D_n(s) = \frac{1}{2\pi} \frac{e^{i(n+1)s} - e^{-ins}}{e^{is} - 1} \Rightarrow \int_{-\pi}^{\pi} D_n(x_0 - x) dx = \frac{1}{2\pi} \sum_{-n}^n \int_{-\pi}^{\pi} e^{ik(x_0-x)} dx = 1$$

(so we have shown that the Fourier series for $f(x) = 1$ converges to 1). Now since we know the convergence of the fourier series for constants

$$S_n(f)(x_0) - f(x_0) = \int_{-\pi}^{\pi} (f(x) - f(x_0)) D_n(x_0 - x) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x) - f(x_0)}{x - x_0} \frac{x - x_0}{e^{i(x-x_0)}} (e^{i(n+1)(x_0-x)} - e^{in(x_0-x)})$$

Now for the crucial part of the argument. We want to show that $S_n(f)(x_0) = A_{n+1} - A_{-n}$ where

$$A_{n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\left(\frac{f(x) - f(x_0)}{x - x_0} \frac{x - x_0}{e^{i(x-x_0)}} \right) e^{i(n+1)x_0} \right] \cdot e^{-ikx}$$

The boxed function is $\in L^2([-\pi, \pi])$, since it is continuous. Thus A_k is a Fourier coefficient of an L^2 function. A_{-n} is similar, but with an opposite sign. By Bessel's Inequality

$$\sum_j |c_j|^2 \leq \|f\|_{L^2}^2 \Rightarrow c_j \rightarrow 0 \Leftrightarrow j \rightarrow \infty, \quad \forall f \in L^2$$

so then $S_n(f)(x_0) - f(x_0) \rightarrow 0$. □

Theorem. If $\mu(A \ominus A_j) \rightarrow 0$ then

$$\int_{A_j} f d\mu \rightarrow \int_A f d\mu$$

Proof. Assume f non-negative. Pass to a subsequence A_k such that $\mu(A_k \ominus A) = \mu(A \setminus A_k) + \mu(A_k \setminus A) \leq 2^{-k}$. Now, it is enough to show that

$$\int_{A \setminus A_k} f d\mu \rightarrow 0$$

Set $B_k = A \setminus A_k$. Then $F = \bigcup B_k$ has finite measure, and so if we define the "remainder" $F_N = \bigcup_{k \geq N} B_k$ then $\mu(F_N) \rightarrow 0$. Now $B_N \subset F_N, \forall N$, so

$$\int_{B_N} f d\mu \leq \int_{F_N} f d\mu$$

So if we can prove that $\int_{F_N} f d\mu \rightarrow 0$ we are done.

Now we write $F = \bigcup_{j \geq 1} G_j$, with $G_j = F_j \setminus F_{j+1}$ and $G_m \cap G_n = \emptyset$. So we have divided F into a countable number of disjoint subsets. So by countable additivity of the integral

$$\int_F f d\mu = \sum_{j=1}^{\infty} \int_{G_j} f d\mu$$

since $f \in L^1$ the left is finite, so the right converges. Thus the "tail" of the sequence goes to 0:

$$\int_{F_N} f d\mu = \sum_{j=N}^{\infty} \int_{G_j} f d\mu \rightarrow 0$$

□

Theorem.

$$\int_{[a,b]} h_1(s) \left(\int_{[a,s]} h_2(x) dx \right) ds = \int_{[a,b]} \left(\int_{[x,b]} h_1(s) ds \right) dx$$

Proof. Linear in h_1 and h_2 . Noting this, we could just prove the above for simple functions and then by linearity again for characteristic functions of a measurable set. Say $h_1 = \chi_A, A \subset [a, b]$. By definition of measurability, A can be approximated by a sequence of sets that are a finite union of disjoint intervals, A_j and $\mu(A \ominus A_j) \rightarrow 0$. So $\int_{A_j} 1 d\mu \rightarrow \int_A 1 d\mu$. And for each A_j we can approximate them by a sequence of intervals. So it suffices to prove the above theorem for intervals. Let $h_1 = \chi_{[a_1, b_1]}$ and $h_2 = \chi_{[c_1, d_1]}$ then we would like that

$$\int_{[a_1, b_1]} \left(\int_{[a_1, s]} \chi_{[c_1, d_1]}(x) dx \right) ds = \int_{[a_1, b_1]} \left(\int_{[x, b_1]} ds \right) \chi_{[c_1, d_1]}(x) dx$$

Let $a = \max(a_1, c_1)$ and $b = \min(b_1, d_1)$ then the above becomes

$$\int_{[a,b]} \left(\int_{[a,s]} dx \right) ds = \int_{[a,b]} \left(\int_{[x,b]} ds \right) dx$$

which is clearly true. □

Apply the above with $h_1 = e^{-inx}$ and $h_2 = f$. Then

$$\int_{-\pi}^{\pi} e^{-ins} \left(\int_{-\pi}^s f dx \right) ds = \int_{-\pi}^{\pi} \left(\int_x^{\pi} e^{-ins} ds \right) f dx$$

The inner integral on the right is just a linear combination of exponentials so the right is 0. And if we define the function

$$g(s) = \int_{[-\pi, s]} f dx$$

then its Fourier coefficients are 0, $\forall n \neq 0$. Then if we adjust by an appropriate constant and let $g(s) = \int_{[-\pi, s]} f dx - C$, where C is such that $c_0(g) = 0$. There is no problem with passing to another function h , with $c_n(h) = 0, \forall n$, which we define as

$$h(y) = \int_{[-\pi, y]} g - C'$$

So we know that $h(y) = 0$ by the pointwise convergence of $S_N(h)$. Thus g is at least constant, since its indefinite integral is 0. Since g is constant, so $\int_{[-\pi, s]} f dx - C$ must be 0 $\forall s$. Now

Theorem.

$$\int_{[a, b]} f d\mu = 0, \quad \forall [a, b] \subset [-\pi, \pi]$$

Proof.

$$\int_{[a, b]} f d\mu = g(b) - g(a) = 0$$

□

And as a consequence of this, and one of the theorems above, the integral of f is 0 on EVERY measurable set. So $f = 0$ almost everywhere. So we have proved completeness.

Theorem. If $f \in L^2([-\pi, \pi])$ then the Fourier series

$$\sum_{n \in \mathbb{Z}} c_n \varphi_n = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} c_n e^{inx} \rightarrow f \in L^2 \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Proof. Bessel's Inequality (φ_n 's are orthonormal). Then

$$\sum_{n \in \mathbb{Z}} |c_n|^2 \leq \|f\|_{L^2}^2 \implies \sum_{n \in \mathbb{Z}} c_n \varphi_n$$

is Cauchy in $L^2([-\pi, \pi])$ because the series converges (its the definition of convergence).

So we know $\sum_n c_n \varphi_n = g$ convergent in $L^2([-\pi, \pi])$. So $c_n(g) = c_n$ by convergence since

$$\langle g, \varphi_n \rangle = \lim_{N \rightarrow \infty} \left\langle \sum_{|k| \leq N} c_k \varphi_k, \varphi_n \right\rangle = c_n$$

$f - g \in L^2([-\pi, \pi]) \subset L^1$, all fourier coefficients vanish are 0, so $f - g = 0$ a.e. in L^2 . □

Periodic function period 2π on \mathbb{R} can be identified with functions on the "circle" $\mathbb{R}/2\pi\mathbb{Z}$ (i.e. $x \equiv y, x - y \in 2\pi\mathbb{Z}$). Can identify $L^2([-\pi, \pi])$ with $L^2(\mathbb{S})$, where $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$

6.2 Proof 2

Theorem. If $\langle f, \varphi_n \rangle = 0, \forall n$ then $f = 0$ almost everywhere.

Theorem. If f is piecewise differentiable then $S_N(f)$ converges uniformly to f .

Proof.

Lemma.

$$c_n \left(\frac{df}{dx} \right) = inc_n(f)$$

Proof.

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{df}{dx} e^{-inx} dx &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left(\frac{d}{dx} (f e^{-inx}) + inf(x) e^{inx} \right) dx \\ &= \frac{1}{\sqrt{2\pi}} (f(\pi) e^{-in\pi} - f(-\pi) e^{in\pi}) + inc_n(f) \end{aligned}$$

if $f(\pi) = f(-\pi)$ (periodicity) then $c_n \left(\frac{df}{dx} \right) = inc_n(f)$ □

and f' is in L^2 as well. So f' fourier series converges, and

$$\sum \left| c_n \left(\frac{df}{dx} \right) \right|^2 = \sum n^2 |c_n|^2 < \infty$$

Now if we can prove that $\sum |c_n(f)|$ converges then we are done. But

$$\sum |c_n(f)| \leq \left(\sum \frac{1}{n^2} \right)^{1/2} \left(\sum n^2 |c_n(f)|^2 \right)^{1/2}$$

□

Let χ_ϵ be the function

$$\chi_\epsilon = \begin{cases} (x-a)/\epsilon, & a \leq x \leq a+\epsilon \\ 1, & a+\epsilon \leq x \leq b-\epsilon \\ (b-x)\epsilon, & b-\epsilon \leq x \leq b \end{cases}$$

(which is basically sort of a function that inclines then stays at 1 and goes back down to 0).

This function χ_ϵ is piecewise differentiable, so then $S_N(\chi_\epsilon)$ converges uniformly to χ_ϵ . Now, $\langle f, S_N(\chi_\epsilon) \rangle = 0$, since $S_N(\chi_\epsilon) = \frac{1}{\sqrt{2\pi}} \sum_{|n| \leq N} c_n(\chi_\epsilon) e^{inx}$, so

$$\langle f, S_N(\chi_\epsilon) \rangle = \left\langle f, \frac{1}{\sqrt{2\pi}} \sum_{|n| \leq N} c_n(\chi_\epsilon) e^{inx} \right\rangle = 0$$

since we are assuming that $\langle f, \varphi_n \rangle = 0, \forall n$. Now, $0 = \langle f, S_N(\chi_\epsilon) \rangle \rightarrow \langle f, \chi_\epsilon \rangle$. Now if we let $\epsilon \rightarrow 0$, then χ_ϵ becomes the characteristic function for $[a, b]$, thus

$$0 = \langle f, \chi_\epsilon \rangle \rightarrow \langle f, \chi_{[a,b]} \rangle = \int_{[a,b]} f d\mu_0$$

This holds for all closed intervals.

6.3 Differential Equations

Try to solve a differential equation such as

$$\frac{d^2 f}{dx^2} + \lambda f = g$$

g, λ given $g \in L^2([-pi, \pi])$ f is to be 2π periodic.

We can guess if $f \in L^2([-pi, \pi])$ then $f = \sum c_n \varphi_n$. Differentiate fourier series term-by-term (its illegal, but who cares)

$$/ \quad \frac{d^2 f}{dx^2} + \lambda f = \sum_{n \in \mathbb{Z}} (\lambda - n^2) c_n \varphi_n = \sum_{n \in \mathbb{Z}} d_n \varphi_n$$

d_n the fourier coefficients of g . So we expect that

$$(\lambda - n^2) c_n = d_n, \quad \text{i.e. } c_n = \frac{d_n}{\lambda - n^2}$$

so if $\lambda = n^2$, then $c_n = d_n/(\lambda - n^2)$. We have $\sum |c_n|^2 < \infty$, $\sum |nc_n|^2 < \infty$, $\sum |n^2 c_n|^2 < \infty$, because $g \in L^2 \Rightarrow \sum |d_n|^2 < \infty$.