

Lecture 13: The General Cauchy Theorem

(Replacing Text 137-148)

Here we shall give a brief proof of the general form of Cauchy's Theorem.
(cf: John D. Dixon, A brief proof of Cauchy's integral theorem, *Proc. Amer. Math. Soc.* 29, (1971) 625-626.)

Definition 1 A closed curve γ in an open set Ω is **homologous to 0** (written $\gamma \sim 0$) with respect to Ω if

$$n(\gamma, a) = 0 \quad \text{for all } a \notin \Omega.$$

Definition 2 A region is **simply connected** if its complement with respect to the extended plane is connected.

Remark: If Ω is simply connected and $\gamma \subset \Omega$ a closed curve, then $\gamma \sim 0$ with respect to Ω . In fact, $n(\gamma, z)$ is constant in each component of $\mathbb{C} - \gamma$, hence constant in $\mathbb{C} - \Omega$ and is 0 for z sufficiently large.

Theorem 1 (Cauchy's Theorem) If f is analytic in an open set Ω , then

$$\int_{\gamma} f(z) dz = 0$$

for every closed curve $\gamma \subset \Omega$ such that $\gamma \sim 0$.

We shall first prove

Theorem 2 (Cauchy's Integral Formula) Let f be holomorphic in an open set Ω . Then

$$n(\gamma, z)f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{1}$$

where $\gamma \sim 0$ with respect to Ω .

Proof: The prove is based on the following three claims.

Define $g(z, \zeta)$ on $\Omega \times \Omega$ by

$$g(z, \zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta-z} & \text{for } z \neq \zeta, \\ f'(z) & \text{for } z = \zeta. \end{cases}$$

Claim 1: g is continuous on $\Omega \times \Omega$ and holomorphic in each variable.

Clearly g is continuous outside the diagonal in $\Omega \times \Omega$. Let (z_0, z_0) be a point on the diagonal and $D \subset \Omega$ a disk with center z_0 . Let $z \neq \zeta$ in D . Then by Theorem 8

$$g(z, \zeta) - g(z_0, z_0) = f'(\zeta) + \frac{1}{2}f_2(z)(z - \zeta) - f'(z_0).$$

So the continuous at (z_0, z_0) is obvious.

For the holomorphy statement, it is clear that for each $\zeta_0 \in \Omega$ the function

$$z \mapsto g(z, \zeta_0)$$

is holomorphic on $\Omega - \zeta_0$. Since

$$\lim_{z \rightarrow \zeta_0} g(z, \zeta_0)(z - \zeta_0) = 0$$

the point ζ_0 is a removable singularity (Theorem 7, p.124), so

$$z \mapsto g(z, \zeta_0)$$

is indeed holomorphic on Ω . This proves Claim 1.

Let

$$\Omega' = \{z \in \mathbb{C} : n(\gamma, z) = 0\}.$$

Define function h on \mathbb{C} by

$$h(z) = \frac{1}{2\pi i} \int_{\gamma} g(z, \zeta) d\zeta, \quad z \in \Omega; \quad (2)$$

$$h(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \Omega'. \quad (3)$$

Since both expressions agree on $\Omega \cap \Omega'$ and since $\Omega \cup \Omega' = \mathbb{C}$, this is a valid definition.

Claim 2: h is holomorphic.

This is obvious on the open sets Ω' and $\Omega - \gamma$. To show holomorphy at $z_0 \in \gamma$, consider a disk $D \subset \Omega$ with center z_0 . Let δ be any closed curve in D . Then

$$\begin{aligned} \int_{\delta} h(z) dz &= \frac{1}{2\pi i} \int_{\delta} \left(\int_{\gamma} g(z, \zeta) d\zeta \right) dz \\ &= \frac{1}{2\pi i} \int_{\gamma} \left(\int_{\delta} g(z, \zeta) dz \right) d\zeta. \end{aligned}$$

For each ζ ,

$$z \mapsto g(z, \zeta)$$

is holomorphic on D (even Ω). So by the Cauchy's theorem for disks,

$$\int_{\delta} g(z, \zeta) dz = 0.$$

Now the Morera's Theorem implies h is holomorphic.

Now we can prove:

Claim 3: $h \equiv 0$, so (1) holds.

We have $z \in \Omega'$ for $|z|$ sufficiently large. So by (3),

$$\lim_{z \rightarrow \infty} h(z) = 0.$$

By Liouville's Theorem, $h \equiv 0$.

Q.E.D.

Proof of Theorem 1: To derive Cauchy's theorem, let $z_0 \in \Omega - \gamma$ and put

$$F(z) = (z - z_0)f(z).$$

By (1),

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} f(z) dz &= \frac{1}{2\pi i} \int_{\gamma} \frac{F(z)}{z - z_0} dz \\ &= n(\gamma, z_0)F(z_0) \\ &= 0. \end{aligned}$$

Q.E.D.

Note finally that Corollary 2 on p.142 is an immediately consequence of Cauchy's Theorem.