

Lecture 2: Exponential function & Logarithm for a complex argument

(Replacing Text p.10 - 20)

For $b > 1$, $x \in \mathbb{R}$, we defined in 18.100B,

$$b^x = \sup_{t \in \mathbb{Q}, t \leq x} b^t$$

(where b^t was easy to define for $t \in \mathbb{Q}$). Then the formula

$$b^{x+y} = b^x b^y$$

was hard to prove directly. We shall obtain another expression for b^x making proof easy.

Let

$$L(x) = \int_1^x \frac{dt}{t}, \quad x > 0.$$

Then

$$L(xy) = L(x) + L(y)$$

and

$$L'(x) = \frac{1}{x} > 0.$$

So $L(x)$ has an inverse $E(x)$ satisfying

$$E(L(x)) = x.$$

By 18.100B,

$$E'(L(x))L'(x) = 1,$$

so

$$E'(L(x)) = x.$$

If $y = L(x)$, so $x = E(y)$, we thus have

$$E'(y) = E(y),$$

It is easy to see $E(0) = 1$, so by uniqueness,

$$E(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots \quad \text{and} \quad E(1) = e.$$

Theorem 1 $b^x = E(xL(b)), \forall x \in \mathbb{R}$.

Proof: Let $u = L(x), v = L(y)$, then

$$E(u + v) = E(L(x) + L(y)) = E(L(xy)) = xy = E(u)E(v),$$

$$E(n) = E(1)^n = e^n,$$

and if $t = \frac{n}{m}$,

$$E(t)^m = E(mt) = E(n) = e^n.$$

so

$$E(t) = e^t, \quad t \in \mathbb{Q}, \quad t > 0.$$

Since

$$E(t)E(-t) = 1,$$

So

$$E(t) = e^t, \quad t \in \mathbb{Q}.$$

Now

$$b^n = E(nL(b))$$

and

$$b^{\frac{1}{m}} = E\left(\frac{1}{m}L(b)\right)$$

since both have same m^{th} power.

$$\left(b^{\frac{1}{m}}\right)^n = b^{\frac{n}{m}} = E\left(\frac{1}{m}L(b)\right)^n = E\left(\frac{n}{m}L(b)\right),$$

so

$$b^t = E(tL(b)), \quad t \in \mathbb{Q}.$$

Now for $x \in \mathbb{R}$,

$$b^x = \sup_{t \leq x, t \in \mathbb{Q}} (b^t) = \sup_{t \leq x, t \in \mathbb{Q}} E(tL(b)) = E(xL(b))$$

since $E(x)$ is continuous.

Q.E.D.

Corollary 1 For any $b > 0, x, y \in \mathbb{R}$, we have $b^{x+y} = b^x b^y$.

In particular $e^x = E(x)$, so we have the amazing formula

$$\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots\right)^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots.$$

The formula for e^x suggests defining e^z for $z \in \mathbb{C}$ by

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots.$$

the convergence being obvious.

Proposition 1 $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.

Proof: Look at the functions

$$f(t) = e^{tz+w}, \quad g(t) = e^{tz} e^w$$

for $t \in \mathbb{R}$. Differentiating the series for e^{tz+w} and e^{tz} with respect to t , term-by-term, we see that

$$\frac{df}{dt} = z f(t), \quad \frac{dg}{dt} = z g(t)$$

and

$$f(0) = e^w, \quad g(0) = e^w.$$

By the uniqueness for these equations, we deduce $f \equiv g$. Thus $f(1) = g(1)$. **Q.E.D.**

Note that if $t \in \mathbb{R}$,

$$e^{it} e^{-it} = 1, \quad \text{and } (e^{it})^{-1} = e^{-it}.$$

Thus

$$|e^{it}| = 1.$$

So e^{it} lies on the unit circle.

Put

$$\begin{aligned} \cos t &= \frac{e^{it} + e^{-it}}{2} = 1 - \frac{t^2}{2} + \cdots, \\ \sin t &= \frac{e^{it} - e^{-it}}{2} = t - \frac{t^3}{3!} + \cdots. \end{aligned}$$

Thus we verify the old geometric meaning $e^{it} = \cos t + i \sin t$. Note that the e^{it} ($t \in \mathbb{R}$) fill up the unit circle. In fact by the intermediate value theorem, $\{\cos t \mid t \in \mathbb{R}\}$ fills up the interval $[-1, 1]$, so $e^{it} = \cos t + i \sin t$ is for a suitable t an arbitrary point on the circle.

Note that $z \mapsto e^z$ takes all values $w \in \mathbb{C}$ except 0. For this note

$$e^z = e^x \cdot e^{iy}, \quad z = x + iy.$$

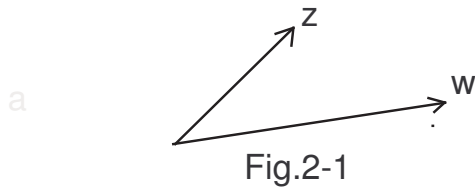
Choose x with

$$e^x = |w|$$

and then y so that

$$e^{iy} = \frac{w}{|w|},$$

then $e^z = w$.



If

$$z = |z|e^{i\varphi}, \quad w = |w|e^{i\psi},$$

then

$$\begin{aligned} zw &= |z||w|e^{i(\varphi+\psi)} \\ &= |z||w|(\cos(\varphi + \psi) + i \sin(\varphi + \psi)), \end{aligned}$$

which gives a geometric interpretation of the multiplication.

From this we also have the following very useful formula

$$(\cos \varphi + i \sin \varphi)^n = e^{in\varphi} = \cos n\varphi + i \sin n\varphi.$$

Thus

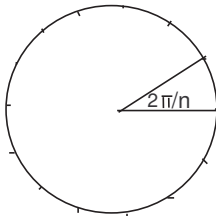


Fig. 2-2

Theorem 2 The roots of $z^n = 1$ are $1, \omega, \omega^2, \dots, \omega^{n-1}$, where

$$\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.$$

Geometric meanings for some useful complex number sets:

$ z - a = r$	\longleftrightarrow	circle
$ z - a + z - b = r, (a - b < r)$	\longleftrightarrow	ellipse
$ z - a = z - b $	\longleftrightarrow	perpendicular bisector
$\{z \mid z = a + tb, t \in \mathbb{R}\}$	\longleftrightarrow	line
$\{z \mid \text{Im}z < 0\}$	\longleftrightarrow	lower half plane
$\{z \mid \text{Im} \left(\frac{z - a}{b} \right) < 0\}$	\longleftrightarrow	general half plane

For x real, $x \mapsto e^x$ has an inverse. This is **NOT** the case for $z \mapsto e^z$, because

$$e^{z+2\pi i} = e^z,$$

thus e^z does not have an inverse. Moreover, for $w \neq 0$,

$$e^z = w$$

has infinitely many solutions:

$$e^x = |w|, \quad e^{iy} = \frac{w}{|w|} \quad \implies \quad x = \log |w|, \quad y = \arg(w).$$

So

$$\log w = \log |w| + i \arg(w)$$

takes infinitely many values, thus not a function.

Define

$$\text{Arg}(w) \triangleq \text{principal argument of } w \text{ in interval } -\pi < \text{Arg}(w) < \pi$$

and define the principal value of logarithm to be

$$\text{Log}(w) \triangleq \log |w| + i \text{Arg}(w),$$

which is defined in slit plane (removing the negative real axis).

We still have

$$\log z_1 z_2 = \log z_1 + \log z_2$$

in the sense that both sides take the same infinitely many values. We can be more specific:

Theorem 3 *In slit plane,*

$$\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2) + n \cdot 2\pi i, \quad n = 0 \text{ or } \pm 1$$

and $n = 0$ if

$$-\pi < \text{Arg}(z_1) + \text{Arg}(z_2) < \pi.$$

In particular, $n = 0$ if $z_1 > 0$.

Proof: In fact, $\text{Arg}(z_1)$, $\text{Arg}(z_2)$ and $\text{Arg}(z_1 z_2)$ are all in $(-\pi, \pi)$, thus

$$-\pi - \pi - \pi < \text{Arg}(z_1) + \text{Arg}(z_2) - \text{Arg}(z_1 z_2) < \pi + \pi + \pi,$$

but

$$\text{Arg}(z_1) + \text{Arg}(z_2) - \text{Arg}(z_1 z_2) = n \cdot 2\pi i,$$

thus

$$|n| \leq 1.$$

If

$$|\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)| < \pi,$$

since

$$|\operatorname{Arg}(z_1 z_2)| < \pi,$$

they must agree since difference is a multiple of 2π .

Q.E.D.