

# Lecture 15: Contour Integration and Applications

(Text 154-161)

## Remarks on Lecture 15

In parts 4 and 5 (p. 154-160) some clarification of the use of the logarithm are called for.

### Example 4 p.159

The relation

$$(-z)^{2\alpha} = e^{2\pi i\alpha} z^{2\alpha}$$

which is crucial for proof deserves an explanation.

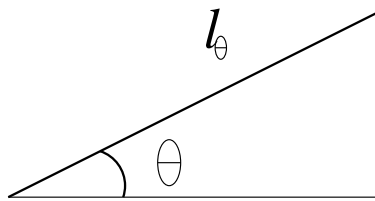


Fig. 15-1

We consider the function

$$\log_{\theta} z = \log |z| + i \arg_{\theta} z$$

in the region  $\mathbb{C} - l_{\theta}$  (the plane with the ray  $l_{\theta}$  removed) where the angle is fixed by

$$\theta < \arg_{\theta} z < \theta + 2\pi.$$

In the problem of computing

$$\int_0^{\infty} x^{\alpha} R(x) dx$$

we consider

$$\log_{-\frac{\pi}{2}}(z)$$

in the plane  $\mathbb{C}$  with the negative imaginary axis removed and us the Residue theorem on the contour in Fig. 4.13. As in the text we arrive at the integral

$$\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2) dz = \int_0^{\infty} (z^{2\alpha+1} + (-z)^{2\alpha+1}) R(z^2) dz.$$

On the right  $z$  belongs to  $(0, \infty)$  and

$$\begin{aligned} \log_{-\frac{\pi}{2}}(z) &= \log |z| + \left(-\frac{\pi}{2} + \frac{\pi}{2}\right) i, & -\frac{\pi}{2} < \arg_{-\frac{\pi}{2}} z < \frac{3\pi}{2}, \\ \log_{-\frac{\pi}{2}}(-z) &= \log |z| + \left(-\frac{\pi}{2} + \frac{3\pi}{2}\right) i \\ &= \log_{-\frac{\pi}{2}}(z) + i\pi, & z > 0. \end{aligned}$$

Thus for  $z > 0$ ,

$$\begin{aligned} (-z)^{2\alpha+1} &= e^{(2\alpha+1)\log_{-\frac{\pi}{2}}(-z)} \\ &= e^{(2\alpha+1)(\log |z| + i\pi)} \\ &= -e^{2\alpha i\pi} z^{2\alpha+1}, \end{aligned}$$

so the last integrals combine to

$$(1 - e^{2\alpha i\pi}) \int_0^{\infty} z^{2\alpha+1} R(z^2) dz.$$

For  $z > 0$  we have from the above

$$\log_{-\frac{\pi}{2}}(z) = \log |z|,$$

so

$$\begin{aligned} \frac{1}{2\pi i} \int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2) dz &= \frac{1}{2\pi i} (1 - e^{2\alpha i\pi}) \int_0^{\infty} x^{2\alpha+1} R(x^2) dx \\ &= -\frac{1}{\pi} e^{\alpha\pi i} \sin \pi\alpha \int_0^{\infty} x^{2\alpha+1} R(x^2) dx. \end{aligned} \tag{1}$$

The left hand side of (1) is the sum of the residues of

$$z^{2\alpha+1} R(z^2) = f(z)$$

in the upper half plane. If

$$R(z^2) = \frac{g(z)}{h(z)},$$

where  $g$  and  $h$  are holomorphic,  $g(a) \neq 0$ , and  $h$  has a simple zero at  $a$ , then

$$\text{Res}_{z=a} f(z) = z^{(2\alpha+1)}(a) \frac{g(a)}{h'(a)}, \tag{2}$$

where

$$z^{2\alpha+1} = e^{(2\alpha+1)\log_{-\frac{\pi}{2}}(z)}.$$

**Example: Exercise 3(g) p.161**

To calculate

$$\int_0^{\infty} x^{\frac{1}{3}} \frac{dx}{1+x^2},$$

we use  $x = t^2$  and arrive at

$$\int_{-\infty}^{\infty} z^{\frac{5}{3}} \frac{dz}{1+z^4}$$

in (1). The poles in the upper half plane are

$$z = e^{i\frac{\pi}{4}} \quad \text{and} \quad z = e^{i(\frac{\pi}{4} + \frac{\pi}{2})}.$$

We use (2) to calculate the residues:

$$\begin{aligned} \operatorname{Res}_{z=e^{i\frac{\pi}{4}}} \left( z^{\frac{5}{3}} \frac{1}{1+z^4} \right) &= z^{\frac{5}{3}} \left( e^{i\frac{\pi}{4}} \right) \frac{1}{4(e^{i\frac{\pi}{4}})^3} \\ &= e^{\frac{5}{3} \log_{-\frac{\pi}{2}}(e^{i\frac{\pi}{4}})} \frac{1}{4(e^{i\frac{\pi}{4}})^3} \\ &= e^{\frac{5}{3}(i\frac{\pi}{4})} \frac{1}{4(e^{i\frac{\pi}{4}})^3} \\ &= \frac{1}{4} e^{-i\frac{\pi}{3}}, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Res}_{z=e^{i\frac{3\pi}{4}}} \left( z^{\frac{5}{3}} \frac{1}{1+z^4} \right) &= z^{\frac{5}{3}} \left( e^{i\frac{3\pi}{4}} \right) \frac{1}{4(e^{i\frac{3\pi}{4}})^3} \\ &= e^{\frac{5}{3} \log_{-\frac{\pi}{2}}(i(-\frac{\pi}{2} + \frac{5\pi}{4}))} \frac{1}{4(e^{i\frac{3\pi}{4}})^3} \\ &= \frac{1}{4} e^{-i\pi}. \end{aligned}$$

Thus (1) gives

$$\frac{1}{4} e^{-i\frac{\pi}{3}} + \frac{1}{4} e^{-i\pi} = -\frac{1}{\pi} e^{\frac{1}{3}\pi i} \sin \frac{\pi}{3} \int_0^{\infty} x^{\frac{5}{3}} \frac{dx}{1+x^4},$$

so

$$\int_0^{\infty} x^{\frac{5}{3}} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{3}}.$$

### Example 5 p.160

The last four lines on the page are a bit misleading because the specific logarithm has already been chosen. So here is a completion of the proof after the equation

$$\int_0^\pi \text{Log}(-2ie^{ix} \sin x) dx = 0.$$

We know (Lecture 2) that

$$\text{Log}(z_1 z_2) = \text{Log}z_1 + \text{Log}z_2, \quad \text{if } -\pi < \text{Arg}z_1 + \text{Arg}z_2 < \pi. \quad (3)$$

Using this for  $z = 2 \sin x$  we get

$$\int_0^\pi \log(2 \sin x) dx + \int_0^\pi \text{Log}(-ie^{ix}) dx = 0. \quad (4)$$

But

$$\text{Log}(-i) = -\frac{\pi i}{2}, \quad \text{Log}e^{ix} = ix \quad (0 < x < \pi),$$

so since  $-\frac{\pi}{2} + x$  is in  $(-\pi, \pi)$ , (3) implies

$$\text{Log}(-ie^{ix}) = -\frac{\pi i}{2} + ix.$$

Now (4) implies the result

$$\int_0^\pi \log \sin \theta d\theta = -\pi \log 2.$$