

Lecture 8: Line Integrals

(Text 101-108)

Remarks on Lecture 8

The following rule (integration by substitution) is often useful.

Theorem 1 *Let $w = \varphi(z)$ be a holomorphic function on a region Ω . Let γ be a curve in Ω , then*

$$\int_{\varphi(\gamma)} f(w) dw = \int_{\gamma} f(\varphi(z))\varphi'(z) dz.$$

Proof: Let γ be given by

$$\gamma : z(t), \alpha \leq t \leq \beta.$$

Then $\varphi(\gamma)$ is given by

$$\varphi : w(t) = \varphi(z(t)), \alpha \leq t \leq \beta.$$

Then LHS equals

$$\int_{\alpha}^{\beta} f(w(t))w'(t) dt,$$

and RHS equals

$$\int_{\alpha}^{\beta} f(\varphi(z(t)))\varphi'(t) dt = \int_{\alpha}^{\beta} f(w(t))w'(t) dt.$$

Q.E.D.

Exercise 3 on page 108.

We have

$$\frac{1}{z^2 - 1} = \frac{1}{2} \left(\frac{1}{z - 1} - \frac{1}{z + 1} \right),$$

so the problem is reduced to computing

$$\int_{\gamma} \frac{1}{z - 1} dz \quad \text{and} \quad \int_{\gamma} \frac{1}{z + 1} dz,$$

where $\gamma : |z| = 2$ is the circle.

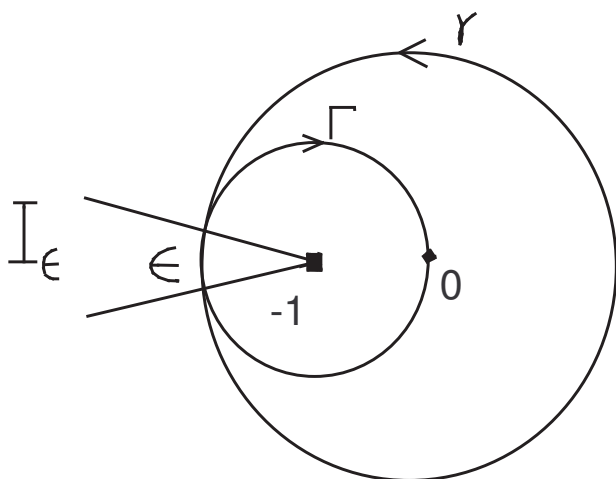


Fig. 8-1

Since $\log(z + 1)$ is holomorphic in the region $\mathbb{C} \setminus I_\epsilon$, where I_ϵ is the shown wedge with vertex -1 and opening of angle ϵ . In this region

$$\frac{d(\log(z + 1))}{dz} = \frac{1}{z + 1}.$$

Letting $\epsilon \rightarrow 0$ we deduce from Theorem p.107 middle

$$\int_{\gamma + \Gamma} \frac{1}{z + 1} dz = 0.$$

Hence

$$\int_{\gamma} \frac{1}{z + 1} dz = 2\pi i.$$

Similarly,

$$\int_{\gamma} \frac{dz}{z - 1} = 2\pi i$$

using the circle $|z - 1| = 1$. Consequently,

$$\int_{\gamma} \frac{dz}{z^2 - 1} = 0.$$

The result is obvious from our substitution theorem because if

$$\varphi(z) = -z,$$

then

$$\varphi(\gamma) = \gamma \quad (\text{including orientation}).$$

So

$$\int_{\gamma} \frac{dz}{z^2 - 1} = - \int_{\gamma} \frac{dz}{z^2 - 1}.$$

More generally we have

Theorem 2 *Let R be a rational function on \mathbb{C} . Then*

$$\int_{\gamma} R(z^2) dz = 0$$

for every circle γ around the origin provided $R(z^2) \neq 0$ on γ .