

Lecture Thirteen: The Heat Equation

We've spent a lot of time concentrating on the laplace equation, but there are other important PDE's. One example is the heat equation, which we will study in this lecture. Consider a function $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ of both time and space. The heat equation is

$$\Delta u = \frac{\partial u}{\partial t}. \tag{1}$$

In this lecture we will prove a gradient estimate and a Harnack inequality for functions satisfying the heat equation on a torus $T^n = S^1 \times S^1 \times \dots \times S^1$, since this turns out to be easier than doing the proof for a ball.

1 A gradient estimate for a torus

Theorem 1.1 *If u is positive and satisfies the heat equation on the cylinder $T^n \times [0, \infty)$ then*

$$\frac{|\nabla u|^2}{u^2} - \frac{1}{u} \frac{\partial u}{\partial t} \leq \frac{n}{2t}. \tag{2}$$

Proof For this proof we will use the notation $g_t = \frac{\partial g}{\partial t}$. Define $f = \log u$, and calculate

$$\begin{aligned} (\Delta - \frac{\partial}{\partial t})f &= \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} - \frac{1}{u} \frac{\partial u}{\partial t} \\ &= -\frac{|\nabla u|^2}{u^2} \\ &= -|\nabla f|^2. \end{aligned}$$

Also define $F = t(|\nabla f|^2 - f_t)$. Note that we actually want to bound $\frac{F}{t}$. We need to estimate $(\Delta - \frac{\partial}{\partial t})F$. Observe that

$$\Delta F = t(\Delta|\nabla f|^2 - \Delta f_t) \tag{3}$$

$$= 2t \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 + 2t \langle \nabla \Delta f, \Delta f \rangle - t \Delta f_t \tag{4}$$

by Bochner. Recall that, for any matrix A_{ij} , $\sum_{i,j} A_{ij}^2 \geq \frac{(\sum_i A_{ii})^2}{n}$ (We saw this in lecture 10, and it essentially because the average of the square is greater than the square of the average). Therefore

$$\Delta F \geq \frac{2t(\Delta f)^2}{n} + 2t \langle \nabla \Delta f, \Delta f \rangle - t \Delta f_t. \quad (5)$$

We also have $\Delta f = -|\nabla f|^2 + f_t = -\frac{F}{t}$, so

$$\Delta F \geq \frac{2F^2}{nt} - 2 \langle \nabla F, \nabla f \rangle - t \Delta f_t. \quad (6)$$

Now work on F_t . Clearly

$$F_t = |\nabla f|^2 - f_t + t(2\nabla f \cdot \nabla f_t) - t f_{tt}. \quad (7)$$

Note that $\Delta f + |\nabla f|^2 = f_t$, so

$$F_t = |\nabla f|^2 - f_t + t(2\nabla f \cdot \nabla f_t) - t(\Delta f + |\nabla f|^2)_t \quad (8)$$

$$= |\nabla f|^2 - f_t - t \Delta f_t. \quad (9)$$

Putting together (6) and (9) we get

$$\left(\Delta - \frac{\partial}{\partial t}\right)F \geq \frac{2F^2}{nt} - 2 \langle \nabla F, \nabla f \rangle - \frac{F}{t}. \quad (10)$$

Pick $s > 0$, and consider the maximum of F on $T^n \times [0, s]$. At the maximum we have $\nabla F = 0$, $\Delta F \leq 0$ and $F_t \geq 0$. Therefore

$$0 \geq \frac{2}{nt} \left(F^2 - \frac{nF}{2}\right), \quad (11)$$

so $F \leq \frac{n}{2}$. Substituting in for F gives

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}, \quad (12)$$

on $[0, s]$. Since this is true for all $s > 0$ we get (12) on $[0, \infty)$, which is what we wanted. ■

2 A Harnack inequality for a torus

Now we'll try to get a Harnack inequality out of this. Pick (x_1, t_1) and (x_2, t_2) with $t_2 \geq t_1$, and let $\eta(t) = (x_2, t_2) + t((x_1, t_1) - (x_2, t_2))$ be the straight line path from one to the other. Then

$$f(x_1, t_1) - f(x_2, t_2) = \int_0^1 \frac{df(\eta)}{ds} ds. \quad (13)$$

Calculate $\frac{df(\eta)}{ds} = \nabla f \cdot (x_1 - x_2) + f_t(t_1 - t_2)$. By inequality (12)

$$f_t(t_1 - t_2) \leq \frac{n}{2t}(t_2 - t_1) - |\nabla f|^2(t_2 - t_1).$$

Together with (13) we get

$$f(x_1, t_1) - f(x_2, t_2) \leq \int_0^1 |\nabla f| |x_2 - x_1| - |\nabla f|^2(t_2 - t_1) + \frac{n(t_2 - t_1)}{2t} ds. \quad (14)$$

The integrand is a quadratic in $|\nabla f|^2$ with negative leading coefficient, so it has a maximum at $|\nabla f| = \frac{|x_2 - x_1|}{2(t_2 - t_1)}$, and

$$f(x_1, t_1) - f(x_2, t_2) \leq \int_0^1 \frac{|x_2 - x_1|}{2(t_2 - t_1)} |x_2 - x_1| - \left(\frac{|x_2 - x_1|}{2(t_2 - t_1)} \right)^2 (t_2 - t_1) + (t_2 - t_1) \frac{n}{2t} ds. \quad (15)$$

We split this up. for the first part

$$\int_0^1 \frac{|x_2 - x_1|}{2(t_2 - t_1)} |x_2 - x_1| - \left(\frac{|x_2 - x_1|}{2(t_2 - t_1)} \right)^2 (t_2 - t_1) ds = \frac{|x_2 - x_1|}{4(t_2 - t_1)}, \quad (16)$$

and for the second

$$\int_0^1 (t_2 - t_1) \frac{n}{2t} ds = (t_2 - t_1) \frac{n}{2} \int_0^1 \frac{1}{t_2 + s(t_1 - t_2)} ds = -\frac{n}{2} \int_{t_2}^{t_1} \frac{1}{v} dv = \frac{n}{2} \log \frac{t_2}{t_1}. \quad (17)$$

Putting these together we have

$$\log u(x_1, t_1) - \log u(x_2, t_2) \leq \frac{|x_2 - x_1|}{4(t_2 - t_1)} + \frac{n}{2} \log \frac{t_2}{t_1}. \quad (18)$$

Taking exponents we get a harnack inequality,

$$\frac{u(x_1, t_1)}{u(x_2, t_2)} \leq \left(\frac{t_2}{t_1} \right)^{n/2} \exp \left(\frac{|x_2 - x_1|}{4(t_2 - t_1)} \right). \quad (19)$$