Lecture 11: More on harmonic functions on a ball

1 Consequences of Hoenig: A new Harnack inequality

The Hoenig inequality we proved last time also gives us a new Harnack inequality. Take u a positive harmonic function on the ball $B_1(0) \setminus \{0\}$. From last time we have

$$\sup_{B_{r/2}(x)} \frac{|\nabla u|}{u} \leq \frac{c}{r} \tag{1}$$

for all $x \in \partial B_r(0)$ with $r \leq 1/2$. Note that $B_{3r/2} \setminus B_{r/2} = \bigcup_{x \in \partial B_r(x)} B_{r/2}(x)$, so

$$\sup_{B_{3r/2} \setminus B_{r/2}} \frac{|\nabla u|}{u} \leq \frac{c}{r}. \tag{2}$$

This has a couple of nice consequences.

Corollary 1.1 *Under the conditions above*

$$\sup_{\partial B_r} u \leq e^{2\pi c} \inf_{\partial B_r} u. \tag{3}$$

Proof Let $f = \log u$, and pick $x, y \in \partial B_r(x)$. There is a path from y to x of length $l \leq 2\pi r$, therefore

$$\begin{aligned}
\log \frac{u(x)}{u(y} &= f(x) - f(y) && (4)\\
&\leq \int_0^l |\nabla f| && (5)\\
&\leq \int_0^l \frac{|\nabla u|}{u} && (6)\\
&\leq \int_0^l \frac{c}{r} && (7)\\
&\leq 2\pi c. && (8)
\end{aligned}$$

Taking exponents we get $\sup_{\partial B_r} u \leq e^{2\pi c} \inf_{\partial B_r} u$ as required. ∎

Corollary 1.2 *Pick w and z in the same ray from the origin with $|w|, |z| \leq 1$. Then*

$$\left(\frac{|w|}{|z|}\right)^c \leq \frac{u(w)}{u(z)} \leq \left(\frac{|z|}{|w|}\right)^c \tag{9}$$

Proof Without loss of generality take $|w| \leq |z|$. Calculate

$$\begin{aligned} \log \frac{u(w)}{u(z)} &\leq \int_{|w|}^{|z|} |\nabla(\log u)| dr & (10) \\ &\leq \int_{|w|}^{|z|} \frac{c}{r} dr & (11) \\ &\leq c \log \frac{|z|}{|w|}, & (12) \end{aligned}$$

and take exponents to get

$$\frac{u(w)}{u(z)} \leq \left(\frac{|z|}{|w|}\right)^c. \tag{13}$$

Note that $\log \frac{u(w)}{u(z)} \geq -\int_{|w|}^{|z|} |\nabla(\log u)|$, and a similar calculation then gives

$$\left(\frac{|w|}{|z|}\right)^c \leq \frac{u(w)}{u(z)}. \quad \blacksquare \tag{14}$$

The two of these together immediately give a third corollary.

Corollary 1.3 *If u is positive and harmonic on $B_1(0) \setminus \{0\}$ then*

$$e^{-2\pi c} \left(\frac{|w|}{|z|}\right)^c \leq \frac{u(w)}{u(z)} \leq e^{2\pi c} \left(\frac{|z|}{|w|}\right)^c. \tag{15}$$

2 Harmonic functions on a pierced disc

It is often useful to consider functions that are harmonic not on an entire disc, but rather on the disc less the center. For example If we have an n-dimensional disc $B_r(0)$ then the functions

$$g = \begin{cases} -\log|x| & \text{if } n = 2 \\ |x|^{2-n} & \text{if } n \geq 3 \end{cases}, \tag{16}$$

which are known as Green's functions, turn out to be very important. However they are clearly only harmonic on $\mathbb{R}^n \setminus \{0\}$. One reasonably straightforward result is the following.

Proposition 2.1 *If u is a harmonic function on $B_1(0) \setminus \{0\}$ with $\frac{u(x)}{g(x)} \to 0$ as $x \to 0$, and v is harmonic on $B_1(0)$ with $u = v$ on $\partial B_1(0)$ then $u = v$ on the entirety of $B_1(0) \setminus \{0\}$.*

Proof Notice that g is everywhere positive. Define $w = u - v$. We must show that $w = 0$ on $B_1(0)$. Let

$$w_+^\epsilon = w + \epsilon g, \tag{17}$$

and

$$w_-^\epsilon = w - \epsilon g. \tag{18}$$

For all $\epsilon > 0$ we can pick δ so that $|w| \le \epsilon g$ on $B_\delta 0$. Therefore w_-^ϵ is negative on $\partial B_1(0)$ and $\partial B_\delta(0)$. Since w_-^ϵ is harmonic on $B_1 \setminus B_\delta$ it takes it's maximum on the boundary, so we know that $w_-^\epsilon < 0$ on $B_1 \setminus B_\delta$. Similarly $w_+^\epsilon > 0$ on $B_1 \setminus B_\delta$. Let ϵ tend to zero, then δ also goes to zero, and we get our result. $\blacksquare$

To finish this lecture we will prove another result similar to the mean value property from lecture one. Let Ω be a subset of $\mathbb{R}^n$ which contains the origin and a small region around it. Take u harmonic on Ω, and $\delta > 0$. By Stokes' theorem

$$0 = \int_{\Omega \setminus B_\delta} g \triangle u = -\int_{\Omega \setminus B_\delta} \nabla u \cdot \nabla g + \int_{\partial(\Omega \setminus B_\delta)} g \nabla u \cdot dS, \tag{19}$$

and

$$0 = \int_{\Omega \setminus B_\delta} u \triangle g = -\int_{\Omega \setminus B_\delta} \nabla g \cdot \nabla u + \int_{\partial(\Omega \setminus B_\delta)} u \nabla g \cdot dS. \tag{20}$$

Taking the difference we have

$$\begin{aligned} 0 &= \int_{\partial(\Omega \setminus B_\delta)} g \nabla u \cdot dS - \int_{\partial(\Omega \setminus B_\delta)} u \nabla g \cdot dS & (21) \\ &= \int_{\partial\Omega} g \nabla u \cdot dS - \int_{\partial B_\delta} g \nabla u \cdot dS - \int_{\partial\Omega} u \nabla g \cdot dS + \int_{\partial B_\delta} u \nabla g \cdot dS. & (22) \end{aligned}$$

Note that g is constant on ∂B_δ, so we can pull it out to get

$$0 = \int_{\partial\Omega} g \nabla u \cdot dS - g(\delta) \int_{\partial B_\delta} \nabla u \cdot dS - \int_{\partial\Omega} u \nabla g \cdot dS + \int_{\partial B_\delta} g \nabla u \cdot dS. \tag{23}$$

Applying Stokes' theorem again we have $\int_{\partial B_\delta} \nabla u \cdot dS = 0$, so

$$0 = \int_{\partial\Omega} g \nabla u \cdot dS - \int_{\partial\Omega} u \nabla g \cdot dS + \int_{\partial B_\delta} u \nabla g \cdot dS. \tag{24}$$

When $n = 2$ we substitute in for g to get

$$\frac{1}{\delta} \int_{\partial B_\delta} u = \int_{\partial\Omega} u \nabla g \cdot dS - \int_{\partial\Omega} g \nabla u \cdot dS, \tag{25}$$

so letting δ tend to zero we have

$$2\pi u(0) = \int_{\partial\Omega} u \nabla g \cdot dS - \int_{\partial\Omega} g \nabla u \cdot dS. \tag{26}$$

When $n > 2$ we get

$$\frac{(n-2)}{\delta^{n-1}} \int_{\partial B_\delta} u = \int_{\partial\Omega} u \nabla g \cdot dS - \int_{\partial\Omega} g \nabla u \cdot dS, \tag{27}$$

so, in the limit,

$$\frac{(n-2)\mathrm{vol}\ \partial B_\delta}{\delta^{n-1}} u(0) = \int_{\partial\Omega} u \nabla g \cdot dS - \int_{\partial\Omega} g \nabla u \cdot dS. \tag{28}$$

From this we can calculate the value of a harmonic function at the origin by it's values on the boundary. Simply by applying a translation this actually allows us to determine the value throughout the interior by knowing the function on the boundary. This also gives another proof that two harmonic functions that are equal on the boundary are equal on the entire set.