NAME:

Pick-up Time: Monday, May 10, 11:00am (in class)

Return Time: Wednesday, May 12, 9:30am (in class)

18.306: Take–Home Quiz # 2

You are NOT allowed to communicate with anyone, other than the instructor, about this quiz while you are taking it. So, work on the test by yourself!

You can use any books or notes, but you are NOT allowed to use calculators, or any software such as Mathematica or Maple for example, in order to solve the problems.

Read the problems CAREFULLY, especially the given HINT(S). Write CONCISE answers, yet JUSTIFY all your steps. If needed, try to use all sheets of paper given to you with the test. Cross out what is not meant to be part of your final answer.

Total # of points: 70.

DO ALL 3 PROBLEMS!

I. (20 pts) [This problem has parts 1, 2, 3.] Consider the usual conservation law in one space dimension x, $-\infty < x < +\infty$, with "density" $\rho = \rho(x,t)$ and "flux" $q = q(x,t) = Q(\rho) + \mu \cdot \rho_{xx}$, where

$$Q(\rho) = \frac{\lambda}{l+1}\rho^{l+1}; \quad l > 0,\ \lambda > 0,\ \mu > 0\,(l: \text{ pure number}).$$

1.(2 pts) <u>Derive</u> a PDE for $\rho(x,t)$ (show all your steps in detail).

2.(6 pts) Consider the stretching transformations (ST) $\tilde{x} = \kappa^{\alpha}\, x$, $\tilde{t} = \kappa^{\beta}\, t$, $\tilde{\rho} = \kappa^{\gamma}\, \rho$ and find relations among the real α, β and γ so that the PDE of part (1) is invariant under these ST ($\kappa > 0$). Construct the form of a similarity solution for $\rho(x, t)$. When would you expect such a solution to hold? (You are NOT asked to derive or solve any equation for the similarity solution at this point.)

3.(a)(5 pts) Now try to solve the PDE of part (1) by use of dimensional arguments that ONLY involve the dimensional quantities ρ, x, t, λ and μ. Define suitable non-dimensional quantities of the form $\rho/A(t)$ and $x/B(t)$; give $A(t)$ and $B(t)$. Construct the form of a similarity solution and compare with your result of part (2) above. Derive an ODE for the similarity solution for arbitrary number l. (You are NOT asked to solve the ODE or provide any conditions at this point.)

3(b)(7 pts) In part 3(a) above, take $l = 2$. Derive a 2nd-order ODE for the similarity solution of 3(a). Justify and give all conditions needed for the solution of this ODE. (You are NOT asked to find the solution.)

II.(20 pts) [This problem has parts 1, 2.] Weakly nonlinear shallow-water waves are governed by the PDE $u_t + \alpha \cdot u u_x + (\beta^2/3)\, u_{xxx} = 0$, $-\infty < x < +\infty$, where α and β are positive constants.

1. (8 pts) Notice that this PDE admits constant solutions, $u = u_0 =$ const. We seek an approximate solution of the form $u = u_0 + u_1(x,t)$ where $|u_1|$ is "sufficiently small." Derive a linear PDE for u_1. Assume a solution in the form $u_1(x,t) = g(kx - \omega t)$ where $g(\theta)$ has period 2π and cannot be a constant. (Here, $\theta \equiv kx - \omega t$.) Show that g is of the form $e^{\pm i(kx-\omega t)}$ and find the dispersion relation $\omega = W(k)$. Give the group and phase velocities.

2. (12 pts) Consider a Fourier superposition of the waves ("wave packet") for the g found in part (1) above and describe it approximately for sufficiently long x and t. Give conditions in terms of the parameters α and β for the validity of your approximation. Define the caustic and provide a graphical and analytical description of the wavepacket near the caustic.

III.(30 pts) [This problem has parts 1, 2.]

1.(15 pts) The two-dimensional (2D) Green's function $G(x, y; x', y')$ for the Laplacian ∇^2 in the rectangle $\Omega = \{0 < x < a,\ 0 < y < b\}$ is defined to be continuous in Ω and satisfy the PDE

$$G_{xx}+G_{yy} = \delta(x-x')\,\delta(y-y'), \quad 0 < x < a,\, 0 < y < b,$$

along with the homogeneous condition $G = 0$ on $\partial\Omega$ ($\partial\Omega$: boundary of Ω). Find G in terms of a suitable series expansion that involves sinusoidal functions in x via using separation of variables.

2. (15 pts) Consider the PDE

$$u_{xx} + u_{yy} + 2u_x + u = \mu \cdot \rho(u), \quad (x, y) \text{ in } \Omega,$$

where $u = u(x, y)$ satisfies the boundary conditions $u(0, y) = u(x, 0) = 0$, $u(a, y) = A = \text{const.}$, and $u(x, b) = B = \text{const.}$; $\rho(u)$ is a nonlinear, bounded function of u and μ is a positive constant.

2(a) (5 pts) Set $u = \phi(x, y) \cdot w(x, y)$ and find a function $\phi(x, y)$ such that the PDE for the new dependent variable $w = w(x, y)$ does NOT contain any w_x and w_y terms.

2(b) (7 pts) Convert the PDE for w that you found in 2(a) to an integral equation. For this purpose, define and find a suitable Green's function; also, define and find a suitable solution $w_0(x, y)$ of the homogeneous PDE for w that results by letting $\rho \equiv 0$. **Hint:** You may use the result of part (1) for the Green function. Also, write $w_0 = w_{0,1} + w_{0,2}$ where $w_{0,1}$ and $w_{0,2}$ each satisfies suitable conditions on $\partial\Omega$; determine $w_{0,1}$ and $w_{0,2}$ separately in terms of series expansions.

2(c) (3 pts) Solve approximately the integral equation of part 2(b) for $|\mu| \ll 1$; your solution should include terms $O(\mu)$ but no higher-order terms.