$\boxed{\text{Ex. 1}}$ $f(x) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}$ $\quad u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4\nu t}} d\xi\, e^{-\xi^2}$ (after a change of variables)

*) We have discontinuous Initial data? Can this discontinuity propagate?
NO, the integral gives us a smooth function (u and derivatives)
$\Rightarrow$ discontinuity data can NOT propagate via diff. system.

*) No sense of (talking about) finite speed of propagation in the solution. Solution appears to "propagate" everywhere instantaneously.
↳ don't really propagate, spread

t=0 t>0

spread, but the area is the same

because diffusion equation comes from conservation law

$$\begin{cases} \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} = 0 \\ q = -\nu g_x \end{cases}$$

March 8, 2004 Lecture 10

Opt. Reading Debnath 1.1 - 1.9
Rev. Session FRI, 4:15 - 5:45
Quiz 1 : MON, March 15

Diffusion eqn: IVP $\begin{cases} u_t = \nu u_{xx} & , -\infty < x < \infty,\ t>0 \\ u(x, t=0) = f(x) \\ u(x,t) \to 0 \text{ as } |x| \to \infty \text{ "sufficiently fast"} \end{cases}$
($u \to 0$, $u_x \to 0$, $u_{xx} \to 0$ etc.)

FT (in x): Solution $u(x,t) = \int_{-\infty}^{+\infty} dx'\, f(x')\, \underbrace{\frac{1}{\sqrt{4\pi\nu t}}\, e^{-\frac{(x-x')^2}{4\nu t}}}_{G(x,x';\,t,0)}$

$G(x,x';t,t') = \frac{1}{\sqrt{4\pi\nu t}}\, e^{-\frac{(x-x')^2}{4\nu(t-t')}}$, $t > t'$, Green's function (Green's function is also useful for non-linear equations)

Feature: Area is conserved $\int_{-\infty}^{+\infty} dx\, u(x,t) = \text{const}$ Why?

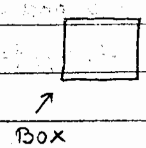
① Diffusion eqn. comes from a conservation law, $g_t + q_x = 0$ ($g \leftrightarrow u$, $q = -\nu u_x$)

$\rho_t + q_x = 0 \quad \Rightarrow \quad u_t = \nu u_{xx}$ (the structure of the PDE)

$\rho = u, \; q = -\nu u_x$

2. Condition for u as $|x| \to \infty$: $u_x \to 0$ as $|x| \to \infty$

this means that the flux $q = -\nu u_x \to 0$ as $|x| \to \infty$



total mass inside the box $\int_{-\infty}^{+\infty} u \, dx$

$\frac{d}{dt}$(total mass) = flux going out but $q = 0$ as $|x| \to \infty$

$\Rightarrow$ mass conserved

Notion of conservation of total mass in diffusion equation:

$$\int_{-\infty}^{+\infty} dx \, u_t = \nu \int_{-\infty}^{+\infty} u_{xx} \, dx$$

$$\frac{d}{dt} \int_{-\infty}^{+\infty} u \, dx = \nu u_x \Big|_{-\infty}^{+\infty}$$

$$= \nu \left[\cancelto{0}{u_x(+\infty, t)} - \cancelto{0}{u_x(-\infty, t)} \right] = 0$$

$$\Rightarrow \int_{-\infty}^{+\infty} dx \, u(x,t) = \text{const}$$

The assumptions we used here

(i) the structure of the PDE

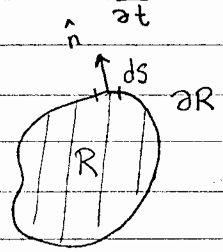
(ii) flux goes to zero at infinity

Let's see what happens if we use the general conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad \text{(1D conservation law)}$$

<u>Ex.</u> In n-dimensions, the form of the conservation law is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot q = 0$$



- area R, boundary ∂R
- the conservation law is true inside R

Let's integrate over the area:

$\rho(\vec{r}, t)$, $\vec{q}(\vec{r}, t)$

from divergence theorem

$$\int_R \frac{\partial \rho}{\partial t} d\vec{r} + \int_R \nabla \cdot \vec{q} \, d\vec{r} = 0 \Rightarrow \frac{d}{dt} \int_R \rho \, d\vec{r} + \int_{\partial R} ds \, \hat{n} \cdot \vec{q} = 0$$

$$\Rightarrow \frac{d}{dt}\underbrace{\int_R \rho\, d\vec{r}}_{\text{total mass in region } R} = -\int_{\partial R} dS\, \hat{n}\cdot\vec{q}$$

Specifically, let's assume

$$\vec{q} = 0 \Big|_{\vec{r}\in\partial R}$$

$$\Downarrow$$

$$\int_R \rho\, d\vec{r} = \text{const}$$

- the diffusion equation is a special case (1D)

In n dimensions:

$$\text{IVP:} \begin{cases} u_t = \nu\nabla^2 u, & \overset{\vec{r}}{\overset{\downarrow}{x}} \in \mathbb{R}^n, \quad t>0 \\ u(\vec{r},t) = f(\vec{r}) \end{cases}$$

Solution (by Fourier transform in all space-variables)

$$u(\vec{r},t) = \int_{\mathbb{R}^n} d\vec{r}\,' f(\vec{r}\,') \frac{1}{(4\pi\nu t)^{n/2}} e^{-\frac{(\vec{r}-\vec{r}\,')^2}{4\nu t}}$$

Next: Optimal Control Theory, Burger equation: nonlinear PDE can be converted by a nonlinear transformation to the diffusion equation.

(diffusion equation: in FT space, we have an ODE)

Nonlinear PDE's convertible to $\overbrace{\qquad}^{\text{Linear PDE's}}$ (Diffusion Equation)

PDE: $\quad u_t - a\nabla^2 u + \underset{\substack{\uparrow \\ \text{nonlinear term}}}{b|\nabla u|^2} = 0 \quad \begin{cases} n \text{ dimensions} + \text{time } t>0 \\ \vec{r} = (x_1, \dots x_n) \quad a, b = \text{const} \end{cases}$

Question: Is there a $\underset{\substack{\uparrow \\ \text{transformation}}}{\text{change}}$ $w = \phi(u)$ that makes this

Nonlinear PDE $\Rightarrow$ a linear PDE

Transformation: $w = \phi(u)$

- $w_t = \phi'(u)\, u_t$
- $\nabla w = \phi'(u)\,\nabla u$
- $\nabla^2 w = \nabla\cdot(\nabla w) = \nabla\cdot(\phi'(u)\nabla u) = \phi'(u)\nabla^2 u + \nabla(\phi'(u))\cdot\nabla u$

$$= \phi'(u)\nabla^2 u + \phi''(u)|\nabla u|^2$$

Apply the PDE:

$$w_t = \phi'(u)\, u_t = \phi'(u)\left[a\nabla^2 u - b|\nabla u|^2\right]$$

$$= a\left[\nabla^2 w - \phi''(u)|\nabla u|^2\right] - b\phi'(u)|\nabla u|^2$$

$$= a\nabla^2 w - \underbrace{\left(a\phi''(u) + b\phi'(u)\right)}_{\substack{\text{we set this to zero (definition of } \phi(u)) \\ \Rightarrow \text{ we will have a linear equation for } w}}|\nabla u|^2$$

- Transformation:

$$a\phi''(u) + b\phi'(u) = 0 \qquad \text{(linear ODE, constant coefficients)}$$

solve for ϕ, any particular solution will do

$$\phi(u) = e^{-\frac{bu}{a}} = w$$

Cole-Hopf transformation (non-linear transformation)

- PDE for w is the diffusion equation: $w_t = a.\nabla^2 w$

IVP: $\begin{cases} u_t - a\nabla^2 u + b|\nabla u|^2 = 0, & \vec{r} \in \mathbb{R}^n,\ t>0 \\ u(\vec{r}, t=0) = g(\vec{r}) \end{cases}$

$\downarrow$

$$\begin{cases} w_t = a\nabla^2 w \\ w(\vec{r}, t=0) = e^{-\frac{bg}{a}} \end{cases} \Rightarrow w(\vec{r},t) = \int_{\mathbb{R}^n} d\vec{r}'\, e^{-\frac{bg(\vec{r}')}{a}} \frac{1}{(4\pi\nu t)^{n/2}} e^{-\frac{(\vec{r}-\vec{r}')^2}{4\nu t}}$$

$$u = -\frac{a}{b}\ln w = -\frac{a}{b}\ln\left\{\int_{\mathbb{R}^n} d\vec{r}'\, e^{-\frac{bg(\vec{r}')}{a}} \frac{1}{(4\pi\nu t)^{n/2}} e^{-\frac{(\vec{r}-\vec{r}')^2}{4\nu t}}\right\}$$

What happens at infinity?

- w is a solution of the linear diffusion equation and an assumption is that $w \to 0$ at $|x_i| \to \infty$

$$w = e^{-\frac{b}{a}u} \Rightarrow \text{Solution for } u \text{ holds if } |u| \to \infty,\ |\vec{r}| \to \infty$$

($+\infty$ or $-\infty$ depends on the sign b/a)

⚠ Be aware that the transformation converts not only the equation but also the conditions.
The difficulty has been moved from the nonlinear PDE to the diffusion equation: solve it with different conditions at infinity.

Example: Burger's equation

$c = c(x,t)$ $\quad c_t + c c_x = \nu c_{xx}$, $\quad c(x,0) = g(x)$

Recall:

conservation law $\rho_t + q_x = 0$ $\Rightarrow \rho_t + \underbrace{Q'(\rho)}_{c(x,t)} \rho_x = \nu \rho_{xx}$

assumed the flux $q = Q(\rho) - \nu \rho_x$

Next, we derived an equation for c

when $Q(\rho)$ is a quadratic function $Q(\rho) = \alpha \rho^2 + \beta \rho + \gamma$

This is the Burger's equation.

$$\boxed{c_t + c c_x = \nu c_{xx}} \quad (1) \quad \rightarrow \quad c_t + \left(\tfrac{1}{2} c^2\right)_x = \nu c_{xx}$$

introduce u: $c = u_x$

$u(x,t) = \int_{-\infty}^{x} dx' \, c(x',t)$ assuming that $c \to 0$ as $|x| \to \infty$ sufficiently fast

$$\Rightarrow (u_t)_x + \left(\tfrac{1}{2} u_x^2\right)_x = \nu (u_{xx})_x$$

integrate $\int^x dx$ $\quad u_t + \frac{1}{2} u_x^2 = \nu u_{xx} + K(t)$

take limit $x \to -\infty$, $u(x,t) = \int_{-\infty}^{-\infty} dx' \, c(x',t) = 0 \Rightarrow K(t) = 0$

$$\boxed{u_t - \nu u_{xx} + \tfrac{1}{2}(u_x)^2 = 0} \quad (2) \quad a = \nu \quad b = \tfrac{1}{2} \quad \text{in 1D}$$

Solution for u: we need initial conditions

IC for u: $\quad g = u_x$, $\quad u = \int_{-\infty}^{x} dx' \, g(x') = h(x)$

Solution for u by Cole-Hopf transformation

$$u(x,t) = -2\nu \ln \left\{ \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} dx' \, e^{-\frac{h(x')}{2\nu}} e^{-\frac{(x-x')^2}{4\nu t}} \right\}$$

$$c(x,t) = +2\nu \left\{ \int_{-\infty}^{+\infty} dx' \, \frac{(x-x')}{2\nu t} e^{-\frac{h(x')}{2\nu}} e^{-\frac{(x-x')^2}{4\nu t}} \right\} \times \left\{ \int_{-\infty}^{+\infty} dx' \, e^{-\frac{h(x')}{2\nu}} e^{-\frac{(x-x')^2}{4\nu t}} \right\}^{-1}$$