May 14, 2004 Lecture 29

Painlevé Equations , $w = w(z)$ $\alpha, \beta, \gamma, \delta = \text{const}$

- $w''(z) = 6w^2 + z$
- $w''(z) = 2w^3 + zw + \alpha$
- $w''(z) = \frac{1}{w}[w'(z)]^2 - \frac{1}{z}w'(z) + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$
- $w''(z) = \frac{1}{2w}[w'(z)]^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$
- $w''(z) = \left(\frac{1}{2w} + \frac{1}{w-1}\right)[w'(z)]^2 - \frac{1}{z}w'(z) + \frac{(w-1)^2}{z}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}$
- $w''(z) = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)[w'(z)]^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w'(z)$

$$+ \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right\}$$

Introduction to Solitons:

Solitary waves: travelling wave , usually localized "hump" that moves without changing shape.

PDE's admitting such solutions:

1. Burger's equation : $u_t + uu_x = u_{xx}$ (from model for turbulence)
2. Thoma's equation (1944):

$$u_{xy} + \alpha u_x + \beta u_y + \gamma u_x u_y = 0 \quad , \quad u = u(x,y)$$

3. KdV eqn. (shallow water waves)

$$u_t \pm 6uu_x + u_{xxx} = 0$$

4. Modified KdV eqn: (plasma physics)

$$u_t \pm 6u^2 u_x + u_{xxx} = 0$$

5. Nonlinear (cubic) Schrödinger eqn. (Gaussian beams in non-linear media)

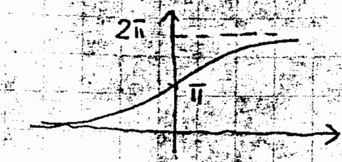
$$iu_t + \nabla^2 u \pm |u|^2 u = 0$$

6. "KP" eqn. $u_{yy} + (u_t + \lambda u_x + 6uu_x + u_{xxx})_x = 0$ $u = u(x,y,t)$

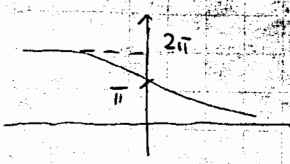
7. Sine Gordon eqn. $u_{xx} - u_{tt} = \sin u$

Two travelling wave solutions:

$$u(x,t) = 4 \arctan \left(e^{\frac{x-vt}{\sqrt{1-v^2}}} \right) \quad : \text{"kink"}$$



$$4 \operatorname{arccot} \left(e^{\frac{x-vt}{\sqrt{1-v^2}}} \right) \quad : \text{"antikink"}$$



Perring & Skyrme (1962)

kink & antikink have a very "simple" structure arctan (exp.)

Suppose :

$$u(x,t) = 4 \arctan [v(x,t)] = 4\theta$$
$\longrightarrow$ PDE for v?

non-trivial solutions for $v(x,t)$: two of them correspond to the kink , antikink but there are many others corresponding to interactions.

2 solitons can collide and then separate same as before except for a phase shift.

- $u_{tt} = \dfrac{4(1+v^2)v_{tt} - 8v_t^2 v}{(1+v^2)^2}$, $u_{xx} = \dfrac{4(1+v^2)v_{xx} - 8v_x^2 v}{(1+v^2)^2}$
- $\sin u = \sin[4\theta] = \dfrac{4\tan\theta\,(1-\tan^2\theta)}{(1+\tan^2\theta)^2} = \dfrac{4v\,(1-v^2)}{(1+v^2)^2}$

$\Rightarrow$ PDE for v highly non-linear

$$(1+v^2)(v_{xx} - v_{tt} - v) - 2v(v_x^2 - v_t^2 - v^2) = 0$$

Try . $v(x,t) = \dfrac{\phi(x)}{\psi(t)}$ historically (guided by $e^{\frac{x-vt}{\sqrt{1-v^2}}}$)

$$\Rightarrow \quad (\phi^2 + \psi^2)\left(\frac{\phi_{xx}}{\phi} + \frac{\psi_{tt}}{\psi}\right) - 2(\phi_x^2 + \psi_t^2) = \phi^2 - \psi^2 \qquad (1)$$

Trick take $\dfrac{d}{dx}$ and $\dfrac{d}{dt}$ of this equation and combine the results

$$\frac{d}{dx}: \quad \frac{\phi^2+\psi^2}{\phi\phi_x}\left(\frac{\phi_{xx}}{\phi}\right)_x - \frac{\phi_{xx}}{\phi} + \frac{\psi_{tt}}{\psi} = 1$$

$$\frac{d}{dt}: \quad \frac{\phi^2+\psi^2}{\psi\psi_t}\left(\frac{\psi_{tt}}{\psi}\right)_t + \frac{\phi_{xx}}{\phi} - \frac{\psi_{tt}}{\psi} = -1$$

Add together:
$$\underbrace{\frac{1}{\phi\phi_x}\left(\frac{\phi_{xx}}{\phi}\right)_x}_{\text{function of } x} + \underbrace{\frac{1}{\psi\psi_t}\left(\frac{\psi_{tt}}{\psi}\right)_t}_{\text{function of } t} = 0$$

! separation of variables can work miraculously for non-linear PDE's too but you have to Apply first non linear transformation.

$$\underbrace{\frac{1}{\phi\phi_x}\left(\frac{\phi_{xx}}{\phi}\right)_x}_{\text{ODE}} = -4\alpha^2 = \underbrace{-\frac{1}{\psi\psi_t}\left(\frac{\psi_{tt}}{\psi}\right)_t}_{\text{ODE}}$$

you solve one, you have the answer of both (similar structure)

$$(\phi')^2 = -\alpha^2\phi^4 + a\phi^2 + b$$
$$(\psi')^2 = +\alpha^2\phi^4 + c\phi^2 + d$$

a, b, c, d - integration constants

ϕ, ψ are elliptic functions (general case)

→ After you differentiate an equation, you loose some information

$x^2 + C \rightarrow 2x$ (lost the constant)

→ After replacing into the equation (1)

$\Rightarrow \{ a - c = 1 \ , \ b + d = 0 \}$ (the constants are not independent)

$+ \ \alpha^2$: ϕ, ψ are elliptic fn. of x, t + 3 parameters

For what value we get the exponentials?

Let $a = m^2 > 0$, $b = n^2$, $c = m^2 - 1$, $d = -n^2$

CASE A: (Exercise) $\alpha = 0$, $n = 0$, $m > 1$ kink or antikink (exponentials)

CASE B: $\alpha = 0$, $m > 1$, $n \neq 0$ (not exponential solutions)

$$\phi(x) = \pm \frac{n}{m} \frac{1}{\cosh(mx + A_1)} \quad (A_1 \rightarrow \text{const})$$

$$\psi(t) = \frac{n}{\sqrt{m^2-1}} \frac{1}{\cosh(\sqrt{m^2-1}\, t + A_2)} \quad (A_2 \rightarrow \text{const})$$

since $u = 4\arctan\left[\frac{\phi(x)}{\psi(t)}\right]$ ⇒ n disappears (right number of constants)

Set $A_1 = A_2 = 0$

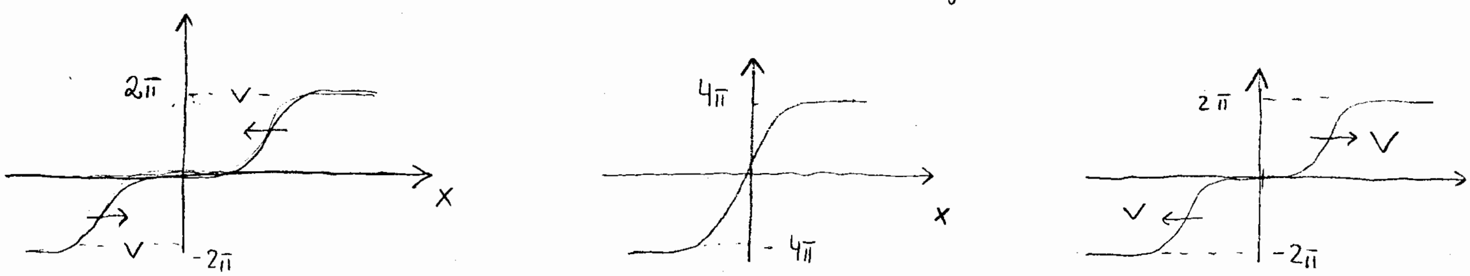
$$u(x,t) = \pm 4 \arctan \left[\frac{V \sinh(mx)}{\cosh(mVt)} \right]$$

How to interpret the solution?

→ take extreme values $-\infty < x, t < +\infty$

$t \to -\infty$ 2 sol. kink, antikink going toward each one

$t \to +\infty$ 2 sol. antikink, kink going away



$t \to -\infty$ | $t = 0$ (merge) | $t \to +\infty$ (phase shift)

General property of solutions of non-linear PDE's:

propagate like pulses (N), collide elastically, emerge with same char. as before possibly with a phase shift.
(intact, without changing shape)

Solitary wave: only one solution

Solitons: many of them that can collide

How to find Soliton solutions?

Inverse Scattering Transform (IST):

Systematic way to find "N-soliton" solutions to certain nonlinear PDE's
many

$\boxed{\text{Idea:}}$ KdV eqn. $u_t - 6uu_x + u_{xxx} = 0$

Gardner, Greene, Kruscal & Miura (1967) $\boxed{u = v^2 + v_x}$ new

$$\left(2v + \frac{\partial}{\partial x}\right)(v_t - 6v^2 v_x + v_{xxx}) = 0$$

If v satisfies $\boxed{v_t - 6v^2 v_x + v_{xxx} = 0}$ $\Rightarrow$ u satisfies KdV

Modified KdV

But look in the other side : suppose you know u $\Rightarrow$ how do we find v

$u = v^2 + v_x$ Ricatti equation

Set $v = \frac{\partial}{\partial x} \ln \psi = \frac{\psi_x}{\psi}$ $\Rightarrow$ $\boxed{\psi_{xx} - u\psi = 0}$ linear ODE

$\downarrow$

$$-\frac{\partial^2 \psi}{\partial x^2} + u\psi = 0$$ Linear Schrödinger eqn.

$$-\psi_{xx} + V\psi = \lambda\psi$$ Sturm-Liouville problem

since $u = V - \lambda$

Certain PDE can be reduced to the linear Schrödinger equation.

Idea IST : $\boxed{\text{Follow}}$ evolution of ψ ("scattering data for ψ") and invert the data to find the potential $V = \lambda + u$

$\downarrow$

find a way to calculate ψ, knowing that u satisfies some PDE's.

In terms of the S.eq this is an inverse problem.

"Old" soliton theory : 70's

How do we know if the IST works for a given PDE? (beforehand)

No answer in terms of theorems.

Conjecture : (Painlevé conjecture)

The IST works for those PDE's that always reduce to ODE's that satisfy the "Painlevé property".

PDE ⇒ ODE

any mean

- similarity way
- traveling way
- sep. of var.
- transformations

Painlevé property (PP)

Digression:

$w = w(z)$, z complex

if $w(z) \sim \dfrac{A}{(z-z_0)^n}$, $z \to z_0$

n ↘ integer

then z_0 is a singular point = pole of order n

Definition:

A critical point, z_0 of the solution $w(z)$ of an ODE is any singular point other than pole.

critical point = singular point that is not pole.

PP:

An ODE satisfies the PP if all its critical points are immovable. i.e. are independent of the integration constants.

Painlevé: What are the possible ODE's 2nd order whose critical points are not movable?

→ Answer: 6 equations (irreducable, any other can be reduced to them)

⇒ If you can reduce your PDE to a Painlevé ODE, then there are soliton solutions.