18.306: Advanced Partial Differential Equations with Applications
M.I.T. Department of Mathematics **Spring 2004**
Homework 6 **Handed out**: Wednesday, 04/28/04
Due: Monday, 05/10/04

The description of the references given in parentheses can be found in the Bibliography for 18.306.

19. The evolution of the surface height $h = h(x,t)$ during the evaporation-condensation process of a material is governed by the PDE: $h_t = \mu \cdot (h_x)^2\, h_{xx}, \; -\infty < x < \infty, t > 0$ and μ: positive constant.

(a) Show that the PDE can be put in the form of a nonlinear diffusion equation with "diffusivity" $\nu = (\mu/3)(h_x)^2$. Argue that the integral $\int_{-\infty}^{\infty} h(x,t)\, dx$ is constant in t if h_x is continuous in x, and $h_x \to 0$ as $|x| \to \infty$.

(b) Solve the given PDE via dimensional arguments by using a similarity solution. For this purpose, impose the condition that $h(x,0) = h(-x,0)$ (ie, even h), and that given, finite amount of material, which is proportional to the integral of h, is confined within a finite region in x with h and h_x being continuous everywhere. **Hint:** You may leave any requisite integration constant(s) in the form of definite integral(s) if you wish.

20. *Dispersion and similarity:*

(a) (Drazin & Johnson, Prob. Q8.1, p. 201.) Long water waves of small amplitude are usually governed by a linear PDE of the (non-dimensional) operational form $u_t + iL(-i\partial/\partial x) \cdot u = 0$, where $L = L(s)$ is an odd function of s that acts on u ($s \equiv -i\partial/\partial x$). Derive the dispersion relation $\omega = W(k)$.**Remark:** For example, for $L(s) = s$ the PDE becomes $u_t + u_x = 0$.

(b) (Drazin & Johnson, Prob. Q1.10, p. 18.) In part (a), take $L(s) = -s^3$; the resulting PDE is the linear KdV equation. If $u(x,0) = f(x)$: given, and u, u_x, $u_{xx} \to 0$ as $|x| \to \infty$, show that for $-\infty < x < +\infty$,

$$u(x,t) = (3t)^{-1/3} \int_{-\infty}^{\infty} dx'\, f(x')\, \mathrm{Ai}\Big(\frac{x - x'}{(3t)^{1/3}}\Big),$$

where $\mathrm{Ai}(z)$ is the Airy function introduced in class.

(c) (Drazin & Johnson, Prob. Q1.13, p. 18.) Consider the (nonlinear) KdV equation, $u_t - 6uu_x + u_{xxx} = 0$. Show that this PDE is invariant under the transformations $x' = \kappa x, t' = \kappa^3 t$ and $u' = \kappa^{-2} u$, where $\kappa > 0$ is an arbitrary parameter. Verify that $t^{2/3}u$ and $xt^{-1/3}$ are also invariant under these transformations. Explain why it is thus reasonable to try the similarity solution $u = -(3t)^{-2/3} g(\eta)$, where $\eta = x(3t)^{-1/3}$. Derive an ODE for $g(\eta)$. **Remark:** This ODE can be solved only numerically in principle, so don't expect to solve it analytically.

Bonus question: By setting $g = \mu\,(dV/d\eta) - V^2$, where $V = V(\eta)$, show that, by a suitable choice of the constant μ, the ODE obtained in (c) for g reduces finally to the 2nd-order ODE $V'' - \eta V - 2V^3 = 0$, assuming that V decays exponentially as either $\eta \to \infty$ or $\eta \to -\infty$.

How is V related to the Airy function in the limit $\eta \to +\infty$ or $\eta \to -\infty$? **Remark:** The ODE for V is known as the "Painlevé equation of the second kind" (but you don't need to know this fact in order to solve the problem).

21. (Carrier & Pearson, Prob. 16.3.1, p. 307.) Consider the PDE $\epsilon\nabla^2 u + u_x + 2u = 0$, where $0 < \epsilon \ll 1$, and $u = u(x, y)$ takes values in the region defined by $0 < x < 1, 0 < y < +\infty$ and satisfies the conditions $u(1, y) = 0 = u(x, 0) = u(x, \infty)$ and $u(0, y) = ye^{-y}$. Identify the boundary layer(s) of this problem and derive an approximation for u by use of boundary-layer theory.