Lecture 12

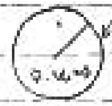
March 15, 2004

Quiz Material 1 : $\begin{cases} \text{Lectures } 1\text{-}11 \\ \text{Homeworks } 1\text{-}2 + \text{Practice Sets } 1\text{-}2 \end{cases}$

Quiz Schedule 1 : $\begin{cases} \text{Pick up : } 4\text{-}5\text{pm} \\ \text{Keep for } 4 \text{ hrs.} \end{cases}$ (5:30 - 6:30 with time conflict)

Pick up Graded Homework 2 after class

Laplace eqn.

Example :  $u = f(\theta)$ $\begin{cases} u: \text{single valued} \\ u: \text{finite} \end{cases}$

Steps

① Identify variables eg (r, θ) :

$$\nabla^2 u = 0 \Leftrightarrow \Big[\underbrace{r^2 \frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r}}_{\mathcal{L}_1(\theta)} + \underbrace{\frac{\partial^2}{\partial \theta^2}}_{\mathcal{L}_2(\theta)} \Big] u = 0$$

separable form

② Try $u(r,\theta) = R(r)\,\Theta(\theta)$ multiplicative separation of variables

③ ODE's for R, Θ solve with $\boxed{\text{homogeneous conditions}}$

$\Theta(\theta)$: periodic , $R(0)$ finite

④ Apply linear superposition : $u = \sum R(r)\,\Theta(\theta)$

all possible ODE solutions

apply remaining conditions $\Rightarrow$ u: unique

Generalization : → operator ex. ∇^2

Linear PDE : $\begin{cases} \mathcal{L}u = 0 & n \text{ dim.} \\ + \text{conditions} \end{cases}$

① Identify variables, say $x_1, x_2, \ldots x_n$

ex. $\mathcal{L}u = 0 \Leftrightarrow [\mathcal{L}_1(x_1) + \cdots + \mathcal{L}_n(x_n)]\, u = 0$

② Separation of variables

Multiplicative: $u = X_1(x_1) \ldots X_n(x_n)$ "product of functions"

TRY

Additive : $u = X_1(x_1) + \ldots X_n(x_n)$ "sum of functions"

the method is heuristic!

either Multiplicative or Additive !

Ex. of additive $u_{tt} - c^2 u_{xx} = 0$ $u = \underbrace{F(x-ct)}_{x_1} + \underbrace{G(x+ct)}_{x_2}$

(This solution is general)

(3) Find ODE's for X_i, $i=1,\dots,n$

Apply homogeneous conditions

(4) Apply $\boxed{\text{linear}}$ supperposition $u = \sum_{\substack{\text{all possible} \\ \text{ODE solutions}}} \binom{\pi}{\Sigma} X_i$, apply remaining conditions

works because the PDE is linear → can be generalized for some non-linear PDE's.

Our general theme PDE(s) ⇒ ODE(s)

"view PDE's as ODE's and consider them soluble"

How many ways to do this if we consider a NONLINEAR PDE's:

(I.) Take the PDE $\boxed{\text{directly}}$ to ODE by "guessing" the form of the solution.

Example: Fisher's PDE $u_t - \nu u_{xx} = \kappa u(1-u/\lambda)$ $\kappa, \lambda > 0$ $u = u(x,t)$

(describes the evolution of some population where we have the effects of diffusion and growth)

$$u_t - \underbrace{\nu u_{xx}}_{\text{diffusion}} = \underbrace{\kappa u\left(1-\frac{u}{\lambda}\right)}_{\text{growth}}$$

"spreading"

~~u is generally > 0~~

~~if u > λ ⇒ growth rate negative~~

"limiting" size of population

The PDE is nonlinear

u has a limiting role on the evolution

"Guess" the solution:

solution is a traveling wave $u = f(x-ct)$

↳ speed of the wave, unknown

$u_t = -c\, f'(\xi)$, $\xi = x-ct$

$u_{xx} = f''(\xi)$ ⇒ from PDE $-cf'(\xi) - \nu f''(\xi) = \kappa f(1-f/\lambda)$

this is an ODE in ξ, can be solved numerically

from "stability analysis" ⇒ c has only certain values that it can take

(II.) Nonlinear PDE (u) goes first to Linear PDE (w) and then try to solve it

1. transformation from $u \xrightarrow{\phi} w$

Example: $u_t - a\nabla^2 u + b|\nabla u|^2 = 0$, $w = \phi(u)$ only u, no space variables

2. potential functions

Example: from fluids last time

3. Hodograph transformations:

in principle, helps convert systems of quasilinear PDE(s) to system of linear PDE(s)

<u>Idea</u>: Switch the role of independent and dependent variables

<u>Example</u>: Steady-state equations of motion of irrotational fluid flow

<u>2D</u>, (u,v) velocity components
$u = u(x,y)$ $c(u,v)$ local sound speed
$v = v(x,y)$ (known function)

$$\begin{cases} [c(u,v)^2 - u^2]u_x - uv(u_y + v_x) + [c(u,v)^2 - v^2]v_y = 0 \\ u_y - v_x = 0 \end{cases}$$

quasi-linear system: linear in the derivatives, non-linear in u, v
<u>Observation</u>: the coefficients do NOT involve x and y.

Instead of $\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$, let $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ <u>Starting point</u>

Express u_x, u_y, v_x, v_y in terms of x_u, x_v, y_u, y_v

<u>2D</u>, $du = u_x\,dx + u_y\,dy$ $dx = x_u\,du + x_v\,dv$
$dv = v_x\,dx + v_y\,dy$ $dy = y_u\,du + y_v\,dv$

$$\begin{cases} du = u_x(x_u\,du + x_v\,dv) + u_y(y_u\,du + y_v\,dv) \\ dv = v_x(x_u\,du + x_v\,dv) + v_y(y_u\,du + y_v\,dv) \end{cases}$$

Set the coeff. in both equations in front of du and dv equal

du: $\begin{cases} 1 = u_x \boxed{x_u} + u_y \boxed{y_u} \\ 0 = v_x \boxed{x_u} + v_y \boxed{y_u} \end{cases}$ dv: $\begin{cases} 0 = u_x \boxed{x_v} + u_y \boxed{y_v} \\ 1 = v_x \boxed{x_v} + v_y \boxed{y_v} \end{cases}$

condition: mapping is one to on

Determinant: $J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ Jacobian $J = \dfrac{\partial(u,v)}{\partial(x,y)}$ $\boxed{J \neq 0}$

$$x_u = \frac{v_y}{J}, \quad y_u = -\frac{v_x}{J}, \quad x_v = -\frac{u_y}{J}, \quad y_v = \frac{u_x}{J}$$

Rewrite the PDE:

$$\begin{cases} [c(u,v)^2 - u^2]\cancel{J} y_v - uv\cancel{J}[-x_v - y_u] + [c(u,v)^2 - v^2]\cancel{J} x_u = 0 \\ -x_v \cancel{J} + y_u \cancel{J} = 0 \end{cases}$$

This system is linear: $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ (u,v) independent variable

Use potential function:

$y_u - x_v = 0$ zero curl $(x,y) = -\nabla_{u,v}\Omega$ potential function

$$x = -\frac{\partial \Omega}{\partial u}, \quad y = -\frac{\partial \Omega}{\partial v}$$

$$[c(u,v)^2 - u^2]\frac{\partial^2 \Omega}{\partial v^2} + uv\left[\frac{\partial^2 \Omega}{\partial u \partial v} + \frac{\partial^2 \Omega}{\partial v \partial u}\right] + [c(u,v)^2 - v^2]\frac{\partial^2 \Omega}{\partial u^2} = 0$$

$$\boxed{[c(u,v)^2 - u^2]\Omega_{vv} + 2uv\,\Omega_{vu} + [c(u,v)^2 - v^2]\Omega_{uu} = 0}$$

4. Legendre transform

in classical mechanics you use this transform to go from Hamiltonian to Lagrangian formulation $(x,y,t) \to (x,y,u,v)$

Idea: Consider as unknown variables the components of the gradient of the unknown function

Example: $\nabla \cdot \left(\dfrac{\nabla u}{[1 + |\nabla u|^2]^{1/2}} \right) = 0$ 2D $u = u(x,y)$

PDE explicitly: $(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0$

(linear in the highest derivative, non-linear in first derivative)

Observations:

- quasilinear 2nd order PDE
- (x,y) do not appear
- involves only derivatives

(two order of derivatives that differ by 1)

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Pick up: Practice Set 3

Homework 3 to be posted TODAY

Opt. Reading $\begin{cases} \text{Kevorkian} & 2.1-2.5 \\ \text{Debnath} & 1.6 \\ \text{Evans} & 2.1-2.4 \end{cases}$

Theme: PDE(s) → ODE(s)

I. Directly:

Characteristics, separation of variables

$\left.\begin{array}{l}\text{traveling waves} \\ \text{similarity solutions (later on)}\end{array}\right\}$ "guess"

II. For nonlinear PDE

convert nonlinear PDE to linear PDE first

1. Transformation $u \to w = \phi(u)$
2. Potential functions
3. Hodograph transform (switch independent and dependent variables)
4. Legendre transformation: view derivatives of unknown functions as independent variables

Example: $(1+u_y^2)u_{xx} - 2\underbrace{u_x u_y u_{xy}}_{\text{cubic term}} + (1+u_x^2)u_{yy} = 0$

but the eq. is QL, 2nd order, no explicit dependence on x, y.

- the derivatives differ by one and only two orders appear.

⇒ View u_x and u_y as independent variables

$\begin{cases} p = u_x(x,y) \\ s = u_y(x,y) \end{cases}$ definition $(x,y) \to (p,s)$ assuming mapping is 1-1 and invertible

Condition: $\begin{vmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{vmatrix} \neq 0 \longrightarrow \begin{cases} x = x(p,s) \\ y = y(p,s) \end{cases}$