April 26, 2004 Lecture 22

Opt. Reading Barenblatt, Chs. 0, 1, 2, 3

Debnath, 8.11 - 8.13

Extra lecture : FRI, Time TBA

Pick up: Handout 8, Summary of last Rev. Session

Dispersion: Slowly varying wave, $u \simeq A(x,t)\, e^{i\theta(x,t)}$

$L \sum_{n=0}^{N} A_n(x,t)$

Steps:

(i) Substitute in PDE, assume

① $A_0, \theta_x, -\theta_t$ coefficients of PDE $= O(1)$ $O(\text{"}\varepsilon\text{"})$ $O(\text{"}\varepsilon\text{"})$

② A, θ_x, θ_t, coeff. of PDE : slowly varying e.g. $\left|\frac{A_x}{A}\right| \ll \frac{1}{\ell_c}$, $\left|\frac{A_t}{A}\right| \ll \frac{1}{\tau_c}$

③ $\left|\frac{A_n}{A}\right| = O(\text{"}\varepsilon^n\text{"})$

(ii) Get 1st-order PDE's for A_0, θ ; PDE ⇒ ODEs!

Can be applied not only to uniform PDE's solvable by FT.

What are l_c, τ_c?

Point of Stationary Phase

Remark: similar issue in the PSP method

$x \to \infty$, $t \to \infty$ means $x \gg \ell_c$, $t \gg \tau_c$

Example: Klein-Gordon equation :

$$u_{tt} - \alpha^2 u_{xx} + \beta^2 u = 0 \qquad \alpha, \beta > 0$$

Dispersion relation: $\omega^2 = \alpha^2 k^2 + \beta^2 \Rightarrow \omega = W(k) = \pm\sqrt{\alpha^2 k^2 + \beta^2}$

$[\partial/\partial t \to i\omega,\ \partial/\partial x \to ik]$

dimension of ω : $\frac{1}{\text{Time}} = \frac{1}{T}$, k: $\frac{1}{\text{Length}} = \frac{1}{L}$

⇒ dimension of α: $\frac{L}{T}$, $\beta = \frac{1}{T}$ ⇒ $\boxed{\tau_c = \frac{1}{\beta},\quad \ell_c = \frac{\alpha}{\beta}}$

Physical meaning: K-G eq. describes oscillations in plasma

$\frac{1}{\tau_c}$: natural frequency of oscillation of free charges (no external forcing)

l_c: wave length of free plasma osillation

- When we say x, t big or small, we compare them with l_c, τ_c.
- Also, l_c & τ_c are usually microscopic, so when we say $x \to \infty$ in SPM, we don't need X to be kilometer...

Stationary-phase method to K.-G. eq.

Start: $I(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} F(k)\, e^{\underbrace{i[kx - W(k)t]}}$

exponent: $kx - W(k)t = kx \mp \sqrt{d^2k^2 + \beta^2}\, t$

here we integrate over k : dimensional
often you have advantage to change to non-dimensional

$$kx \mp \sqrt{d^2k^2+\beta^2}\, t = \beta\left[\frac{k}{\beta}x \mp \sqrt{\left(\frac{dk}{\beta}\right)^2+1}\; t\right]$$

new variable: $\xi = \frac{dk}{\beta}$ non-dimensional

$$\text{exponent} = \underbrace{\beta t}_{\text{dimensionless}}\left[\xi \underbrace{\frac{x}{dt}} \mp \sqrt{\xi^2+1}\right]$$

For the stationary-phase to be valide:

$\boxed{\beta t \gg 1} \Rightarrow t \gg \frac{1}{\beta} = \tau_c$ (i) big parameter outside the exponent

(ii) everything else to be of order 1

$\boxed{\frac{x/t}{d} = O(1)} \Rightarrow \frac{x}{t} = O(d)$

combining both we get $x \gg d/\beta = l_c$

Similarity Solutions: Particular solutions to PDE which are approximate by nature.

↳ satisfy the PDE, but what about conditions?

by nature, they can not be exact.

Ex. Diffusion equation

$$\begin{cases} u_t = \nu u_{xx} & -\infty < x < \infty, \; t > 0 \\ u(x, t=0) = f(x) & \\ u \to 0, & |x| \to \infty \end{cases}$$

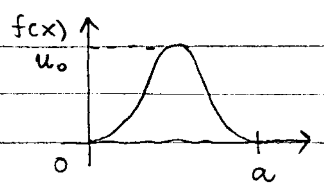
$[x] = L$, $[t] = T$ $\Rightarrow$

- $[\nu] = L^2/T$ (comes from PDE)
- for u, can not find the dimension from the PDE (it's homogeneous)

all we know is $[u] = [f]$ (not the case for non-linear PDE's)

Recall: $$u(x,t) = \int_{-\infty}^{+\infty} dx' \, f(x') \, \frac{e^{-(x-x')^2/(4\nu t)}}{\sqrt{4\pi\nu t}}$$

Assume initial data $f(x)$ extends only over interval of length a



$[u_0] = [u]$, $[a] = L$

$$u(x,t) = \int_0^a dx' \, f(x') \, \frac{e^{-\frac{(x-x')^2}{4\nu t}}}{\sqrt{4\pi\nu t}}$$

complicated solution: let's try to approximate it?

Q: What is u for sufficiently large times?

"$t \to \infty$", $[\nu] = L^2/T$, $\Rightarrow$ $[t\nu] = L^2$ $\Rightarrow$ $\frac{a}{\sqrt{\nu t}}$ = Dimensionless

⇓

$\frac{a}{\sqrt{\nu t}} \gg 1$ but we want also $\underbrace{\frac{x}{\sqrt{\nu t}} \gtrsim O(1)}_{\text{makes life difficult but interesting}}$

Zoom into the exponent:

$$e^{-\frac{(x-x')^2}{4\nu t}} \qquad \frac{x^2}{4\nu t} + \cancel{\frac{x'^2}{4\nu t}} - 2\,\frac{x}{\sqrt{4\nu t}}\,\frac{x'}{\sqrt{4\nu t}}$$

x' do not exceed a $\Rightarrow$ $\frac{x'^2}{4\nu t} \ll 1$

leading term is: $x^2/4\nu t$

Approximate solution $\frac{a}{\sqrt{\nu t}} \ll 1$ $\quad \frac{x}{\sqrt{\nu t}} \gg O(1)$

$$u(x,t) \approx \frac{e^{-x^2/4\nu t}}{\sqrt{4\pi\nu t}} \int_0^a dx'\, f(x')$$

- time is suff. long so that the length of the data appears as small but not so long that x appears small → intermediate time
- u sees only the integral of the data
  u is "blind" to the fine details of initial data

How can we find such an approximate u from the beginning without using the exact solution? (carrive at the appr. solution without solving the PDE exactly by FT)

Method A (does not always work): Dimensional Analysis

Step 1 (i) Form independent, non-dimensional groups of parameters or independent variables

$$\frac{x}{\sqrt{\nu t}}, \quad \frac{a}{\sqrt{\nu t}} \qquad \left(\frac{x}{a} \text{ not independent}\right)$$

(ii) u is a function of these groups of parameters.

$$u = g\Big(\underbrace{\frac{x}{\sqrt{\nu t}}}_{\eta}, \underbrace{\frac{a}{\sqrt{\nu t}}}_{\xi}\Big)$$

Now take "$t \to \infty$", $\frac{a}{\sqrt{\nu t}} \ll 1$, $\frac{x}{\sqrt{\nu t}} = O(1)$

Assume $\underset{\llcorner\, \eta \to 0^+}{g(\eta,\xi)} \approx \eta^{\overset{\text{to be found}}{\beta}} h(\xi)$

$\Rightarrow \quad u \approx \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} h(\xi) \,, \quad \xi = \frac{x}{\sqrt{\nu t}} \qquad$ PDE $\Rightarrow$ ODE in ξ !

PDE:

- $u_{xx} = \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} \frac{1}{\nu t} h''(\xi)$
- $u_t = \left(\frac{a}{\sqrt{\nu}}\right)^{\beta} (-\beta/2) \left(\frac{1}{\sqrt{t}}\right)^{\beta} \frac{1}{t} h(\xi) + \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} h'(\xi) \frac{x}{\sqrt{\nu}} \left(-\frac{1}{2}\right) \frac{1}{\sqrt{t}} \frac{1}{t}$

$$= \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} \left(-\frac{\beta}{2t}\right) h(\xi) + \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} \xi h'(\xi) \left(-\frac{1}{2t}\right)$$

Replace in PDE :

$$-\frac{\beta}{2t} h(\xi) - \frac{\xi}{2t} h'(\xi) = \nu \frac{1}{\nu t} h''(\xi)$$

$\Rightarrow \quad \boxed{h''(\xi) + \frac{\xi}{2} h'(\xi) + \frac{\beta}{2} h(\xi) = 0} \qquad$ ODE in ξ

How do we find β ?
What does u keeps from initial data?

$$u_t = \nu u_{xx} \Rightarrow \int_{-\infty}^{+\infty} dx\, u_t = \nu \int_{-\infty}^{+\infty} dx\, u_{xx}$$

$$\Rightarrow \frac{d}{dt} \int_{-\infty}^{+\infty} u\, dx = \nu \left[\cancelto{0}{u_x}\Big|_{x=+\infty} - \cancelto{0}{u_x}\Big|_{x=-\infty} \right] = 0$$

no matter how approximate the solution is, it has to remember the integral of the IC:

$$\Rightarrow \int_{-\infty}^{+\infty} u(x,t)\, dx = \text{const} = \int_{-\infty}^{+\infty} f(x)\, dx$$

→ For PDE coming from conservation laws, the area is preserved.
→ Different approach for different PDE's

Apply the idea:

$$u = \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} h(\xi)$$

$$\int_{-\infty}^{+\infty} dx\, u = \int_0^a dx\, f(x) = \int_{-\infty}^{+\infty} dx \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} h(\xi)$$

$$= \sqrt{\nu t} \int_{-\infty}^{+\infty} d\xi \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} h(\xi)$$

$$= \sqrt{\nu t} \left(\frac{a}{\sqrt{\nu t}}\right)^{\beta} \underbrace{\int_{-\infty}^{+\infty} d\xi\, h(\xi)}_{\text{number} = C}$$

$$\underbrace{\int_0^a dx\, f(x)}_{\text{do not depend on } t} = \underbrace{\frac{a^{\beta}}{(\sqrt{\nu t})^{\beta-1}}}_{\text{should not dependent on } t} C \quad \Rightarrow \beta = 1$$

April 28, 2004 | Lecture 23

Extra lecture: FRI 4-5:30

Return: Hmwk 5 (Place in envelope)

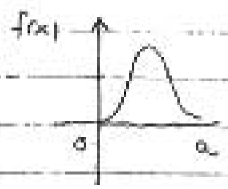
Pick up: Graded Hmwk 4

New Hmwk 6

<u>Review</u>: Example PDE
$$\begin{cases} u_t = \nu u_{xx} \\ u(x, t=0) = f(x) \\ u \to 0 \text{ "fast" as } |x| \to \infty \end{cases}$$

From exact solution:

$$u(x,t) \approx \frac{e^{-x^2/4\nu t}}{\sqrt{4\pi\nu t}} \int_0^a dx\, f(x)$$



conditions $\frac{a}{\sqrt{\nu t}} \ll 1 \qquad \frac{x}{\sqrt{\nu t}} \geq O(1)$

Get u under same conditions without solving PDE exactly.