18.306: Advanced Partial Differential Equations with Applications
M.I.T. Department of Mathematics **Spring 2004**
Homework 1 **Handed out**: Wednesday, 02/11/04
Due: Wednesday, 02/25/04

The description of the references in parentheses can be found in the Bibliography for 18.306.

1. (a) Find the general solution $u(x, y)$ of the PDE

$$u_x + 2x(e^{-x^2} - y)\, u_y = x^2.$$

   (b) Describe (with the help of a sketch) the region R of the xy-plane in which $u(x, y)$ is determined by prescribed values of u along the line segment that connects the points $(0, 0)$ and $(0, 1)$. If $u = 1 + y$ on this line segment, find $u(x, y)$ in R.

2. This problem is an amusing application of PDE to traffic flow. Assume that the car speed V is given in terms of the car density ρ by the equation

$$V = V(\rho) = V_0\Big(1 - \frac{\rho}{\rho_m}\Big),$$

   where ρ_m is known and $0 \leq \rho \leq \rho_m$. After a long red light at $x = 0$, the road is empty for $x > 0$, and is full of stationary, bumper-to-bumper cars for $x < 0$. At time $t = 0$, the light turns green.

   (a) Sketch the characteristics. **Hint:** You may fill the xt-plane with characteristics by considering first a smooth function $\rho(x, 0)$ that is very steep near $x = 0$, and then let the slope at $x = 0$ become infinite.

   (b) How long after the light turns green will a car initiallly at $x = -a$ $(a > 0)$ start moving?

   (c) Where will this car be when its speed becomes $V_0/2$?

   (d) Where will this car be when its speed becomes λV_0 $(0 < \lambda < 1)$, and how long after the light turns green will it get there? **Hint:** Denote the position of the car as a function of time by $x_a(t)$, and note that $x_a(t)$ satisfies the differential equation $dx_a/dt = V(x_a, t)$.

3. (Carrier & Pearson, p. 102) Describe the solution $u = u(x, y, z)$ of the PDE $xu_x + yu_y + uu_z = 0$, where u satisfies the condition $u(x, y, 0) = xy$ for $x > 0$, $y > 0$.