$\Rightarrow$ the solution is $\boxed{\varphi(x) = \varphi_0 \cdot sn\left(\sqrt{\omega - \frac{a}{2}\varphi_0^2}\, x \,;\, \kappa\right)}$

take $x = L/2$

$$\varphi(L/2) = \varphi_0 = \varphi_0 \; sn\left(\sqrt{\omega - \frac{a}{2}\varphi_0^2}\,\frac{L}{2} \,;\, \kappa\right)$$

$$\Downarrow$$

$$1 = sn\left(\sqrt{\omega - \frac{a}{2}\varphi_0^2}\,\frac{L}{2} \,,\, \kappa\right)$$

$$\Downarrow$$

we can get ω but def of ω depends on φ_0, the value at the pick

April 5, 2004 — Lecture 16

Pick up: Handouts 7, 8, Solutions to Hmwk 3
Hmwk 4, Problem Set 4

Pick up: Graded Test 4-6 pm

Opt. Reading: Kevorkian 4.1-4.4, Debnath 1.11

Natural frequencies:

Ex 1 Linear Schrödinger eqn. (LSE)

$$\begin{cases} i\Psi_t = -\Psi_{xx} + \cancelto{0}{V(x,t)}\,\Psi \\ \Psi(0,t) = 0 = \Psi(L,t) \end{cases}$$ Homogeneous problem

Try $\Psi(x,t) = e^{-i\omega t}\varphi(x) \Rightarrow \begin{cases} \varphi''(x) + \omega\phi(x) = 0, & 0 < x < L \\ \phi(0) = 0 = \phi(L) \end{cases}$

$\Rightarrow \phi(x) = A \sin(\sqrt{\omega}x)$ — arbitrary constant

Zeros correspond to $\omega_n = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, \ldots$

ω_n ! independent of A

constraint does not come from the homogeneous equation → Const. A : $\int_0^L dx\,|\Psi|^2 = 1 \Rightarrow |A|^2 = \frac{2}{L}$

Ex. 2 Nonlinear Schrödinger eqn. (NLSE)

$$\begin{cases} i\Psi_t = -\Psi_{xx} + a|\Psi|^2\Psi \\ \Psi(0,t) = 0 = \Psi(L,t) \end{cases}$$

Try $\Psi = e^{-i\omega t}\phi(x) \Rightarrow \begin{cases} \omega\phi(x) = -\phi''(x) + a|\phi|^2\phi, & 0<x<L \\ \phi(0) = 0 = \phi(L) \end{cases}$

"we try to find natural frequencies"

Assummed: $\begin{cases} \omega \text{ in } \{\omega_n\} , & n = 1,2,\dots \\ \phi \text{ real and } \phi'(x) > 0 \text{ for : } 0 < x < L/2 & \text{for } \omega = \omega_1 \end{cases}$

1) ω is still discrete

2) ϕ is real and $\underbrace{\phi' > 0 \text{ for the lowest } \omega_1}$: i.e same symmetry as for the linear problem

$$\Rightarrow \phi(x) = \phi(L/2) \; sn\left(\sqrt{\omega - a/2\, \varphi(L/2)^2}\; x \,;\, K\right) , \quad K^2 = \frac{a/2}{\omega - \frac{a}{2}\varphi(L/2)^2} \;;\; \omega = \omega_1$$

$\phi'(L/2) > 0$; elliptic function

$$\omega_1 : \; sn\left(\sqrt{\omega - \frac{a}{2}\varphi(L/2)^2}\; \frac{L}{2} \,;\, K\right) = 1 \Rightarrow \omega_1 \text{ depends on } \varphi(L/2)$$

A

! we cannot factor it out as in linear problem

bound in ω:
$\omega \geq \frac{a}{2}\varphi(L/2)^2$

→ it is convenient for us to think of a as a small parameter, but it turns out that the NLSE is soluble exactly in elliptic functions.

More general treatment of the nonlinear problem for the SE (NLSE)

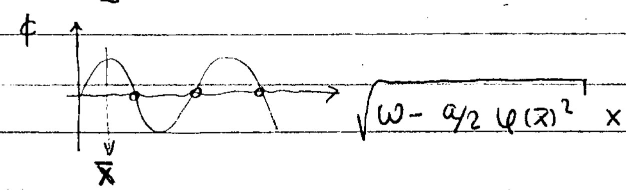
Let's assume that for any $\omega > 0$: ϕ is real. (doesn't have to be $\frac{L}{2}$)
ϕ vanishes at both ends, then there exists $\bar{x}$: $\phi(\bar{x}) = 0$
by solving the ODE we can find that ϕ takes the form

$$\phi(x) = \phi(\bar{x})\; sn\left(\sqrt{\omega - \frac{a}{2}\phi(\bar{x})^2}\; x \,;\, K^2\right) \quad \text{where } K^2 = \frac{a/2}{\omega - a/2\,\phi(\bar{x})^2}$$

(all the derivations from last time hold just replace $L/2 \to \bar{x}$)
This solution can be extended for all $0 < x < L$

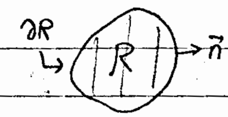
Can we get a set of discrete natural frequencies from knowing that the solution can be expressed in elliptic functions?

$sn\left(\sqrt{\omega - \frac{a}{2}\varphi(\bar{x})^2}\; x \,;\, K^2\right)$ has discrete zeros and is periodic



General "eigenvalue problem" for linear PDEs :
most theorems are about linear problems

$$\begin{cases} \mathcal{M}u = \mu \mathcal{N} u \quad (\mu \text{ const}) & \mathcal{M},\mathcal{N} \text{ are linear operators in n-dim space} \\ & \vec{r} \text{ in region } R \\ u: \text{satisfies homogeneous bc's on } \partial R \\ (\text{eg } au + b\frac{\partial u}{\partial n} = 0 \text{ on } \partial R \end{cases}$$



Definition: M,N are called self-adjoint operators in R if for any v, w that satisfy the given homogeneous bc's on ∂R

$$\int_R d\vec{r}\; vMw = \int_R d\vec{r}\; wMv$$

and

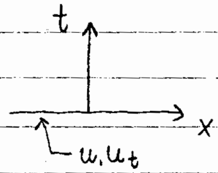
$$\int_R d\vec{r}\; v\,Nw = \int_R d\vec{r}\; w\,Nw \quad .$$

Theorem: If M and N are self-adjoint then
(i) μ takes values only from a discrete set $\{\mu_n\}$ $n = 1, 2, 3 \ldots$
(ii) u also takes values in a discrete set
$\{u_n\}$ corresponding to $\{\mu_n\}$;
$\{u_n\}$ form a complete orthogonal set in R

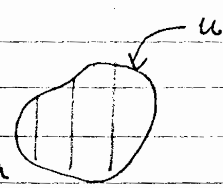
The Sturm-Liouville problem is a special case of this problem.
(1-dim and second order operators)

Classification of 2nd order equations

HYPERBOLIC
Wave eqn.
Initial data
(Cauchy)

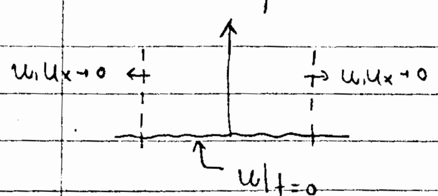


Laplace eqn.
boundary data



ELLIPTIC

Diffusion eqn. $u_t = \nu u_{xx}$



$\begin{pmatrix} \text{FT} \\ \text{in } x \end{pmatrix}$ PARABOLIC

Any 2nd order quasi-linear PDE

$$\begin{cases} Au_{xx} + 2Bu_{xy} + Cu_{yy} + D = 0 & u = u(x,y) \\ A, B, C, D \text{ are functions of } x, y, u, u_x, u_y \end{cases}$$

HYPERBOLIC : if $\Delta = B^2 - AC > 0$

ELLIPTIC : if $\Delta = B^2 - AC < 0$

PARABOLIC: if $\Delta = B^2 - AC = 0$

Questions :

* Δ can change sign depending on the region you are. How do you pose well your problem?
* why is this classification possible?

∘ Wave eqn. $u_{tt} - c^2 u_{xx} = 0$

$A = 1, \quad B = 0, \quad C = -c^2 \quad \Rightarrow \quad \Delta = c^2 > 0$

this is a hyperbolic equation

What this means? Let's get an indication:

Try $u = e^{px+qt} \quad \Rightarrow \quad q^2 - c^2p^2 = 0 \quad \Rightarrow \quad q = \pm cp$

• if $p = i\underbrace{\xi}_{\text{real}} \quad \Rightarrow \quad q$: imaginary too

meaning: if you have something oscillatory in x, the PDE forces that you have oscillations in t too.

$\Rightarrow$ the frequency content of the initial data will propagate along t

$\Rightarrow$ any discontinuities will propagate too.

∘ Laplace eqn. $u_{xx} + u_{yy} = 0$

$A = 1, \ B = 0, \ C = 1 \quad \Rightarrow \quad \Delta = -1 < 0$

this is an elliptic equation.

Do the same trick:

Try $u = e^{px+qy} \quad \Rightarrow \quad q^2 + p^2 = 0 \quad \Rightarrow \quad q = \pm ip$

• if $p = i\xi$ (pure imaginary) $\Rightarrow$ $q \lessgtr 0$ real

meaning: if you have an oscillatory solution in x; it can decay or grow in y.

$\Rightarrow$ initial data oscillatory in x do not propagate along y

Worst, the IVP for Laplace equation is ill-posed.

$\Rightarrow$ discontinuities do not survive neither.

- Diffusion eqn. $u_t = \nu u_{xx}$

$A=\nu$, $B=0$, $C=0$ $\Rightarrow$ $\Delta = 0$

This is a parabolic equation

Try $u = e^{px+qt}$ $\Rightarrow$ $q = \nu p^2$

- if $p = i\xi$ $\Rightarrow$ $q < 0$ real negative

meaning: ~ elliptic but only decay in y is possible.

More sophisticated /rigorous approach :

- meaning of classifications of PDEs

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + D = 0$$

(We suppose also that : $A = A(x, y, \cancel{u}, u_x, u_y)$)

Make a system:

$v = u_x, \quad w = u_y$

$$\begin{cases} Av_x + 2Bv_y + Cw_y = -D \\ w_x = v_y \end{cases}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}}_{M_1} \begin{pmatrix} w_x \\ v_x \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 \\ C & 2B \end{pmatrix}}_{M_2} \begin{pmatrix} w_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ -D \end{pmatrix}$$

Characteristic directions : $\begin{cases} \delta x = \alpha \varepsilon \\ \delta y = \beta \varepsilon \end{cases}$

$$\det(\alpha M_1 - \beta M_2) = 0 \Rightarrow \det\begin{pmatrix} \alpha & \beta \\ -\beta C & \alpha A + \beta 2B \end{pmatrix} = 0$$

$$\alpha(\alpha A - \beta 2B) + \beta^2 C = \alpha^2 A + 2\beta\alpha B + \beta^2 C = 0$$

$$\left(\frac{\alpha}{\beta}\right)^2 A - 2\left(\frac{\alpha}{\beta}\right) B + C = 0 \quad (\text{suppose } \beta = 0)$$

Quadratic equation in $\frac{\alpha}{\beta}$: $\Delta = B^2 - AC$ $\begin{cases} > 0 & \text{two real roots (Hyperbolic)} \\ < 0 & \text{complex, non real (Elliptic)} \\ = 0 & \text{double real root (Parabolic)} \end{cases}$