

You are NOT required to return solutions for this set. However, these problems may be given in Quiz 1. Problems 1-3 serve as a review of ordinary differential equations (ODE). The description of the references given in parentheses can be found in the Bibliography for 18.306, with the exception of the reference for Problem 3 of this set.

1. Find all solutions of the initial value problem (IVP)

$$x^2 u''(x) + x u'(x) - u(x) = 0, \quad u(1) = 0, \quad u'(1) = A; \quad x > 1.$$

**Hint:** This ODE is the equidimensional equation (see Handout 3).

2. (a) Find the solution of the IVP

$$u'(t) + q(t) u(t)^2 + p(t) u(t) = 0, \quad u(0) = u_0 > 0; \quad t \geq 0,$$

where  $p(t)$  and  $q(t)$  are known functions of  $t$ .

**Hint:** See Handout 3; you may use  $v = 1/u$  as a new dependent variable.

(b) Consider the special case  $q(t) = -\nu$  and  $p(t) = \mu$ , where  $\mu$  and  $\nu$  are *positive* constants ( $\mu > 0$  and  $\nu > 0$ ). Find the relation that  $u_0, \mu$  and  $\nu$  should satisfy so that  $u(t)$  develops a *singularity at finite time*  $t = t_s > 0$ , i.e., so that  $u \rightarrow \infty$  as  $t \rightarrow t_s$ . Determine  $t_s$  in terms of  $u_0, \mu$  and  $\nu$ .

**Remark (not needed to solve the problem!):** The development of singularities at finite time is a pathological feature of many *nonlinear* differential equations (here an ODE). In fact, an important “million dollar” problem in fluid dynamics is to establish whether solutions of the celebrated three-dimensional Navier-Stokes PDE, which describe the motion of a viscous fluid, ‘blow up’ or not in time.

3. (Prob. 1.8.3, pp. 14, 15 in G. F. Carrier & C. E. Pearson, “Ordinary Differential Equations”, Blaisdell, 1968.) As you know, ODE are often used to model physical phenomena. This problem provides one such example. Occasionally, a meteorite, an artificial satellite, or some other object enters the Earth’s atmosphere on a path approximately vertical downward. The force,  $F$ , that the atmosphere exerts on such a rapidly moving object is known to be approximated by  $F = \rho V^2 AC/2$ , where  $\rho$  is the atmospheric density,  $V$  is the speed of the object,  $A$  is its projected cross-sectional area, and  $C$  is a constant that depends on the geometry; for blunt objects  $C$  can be taken equal to 2,  $C = 2$ . Let us consider objects that only lose a negligible amount of their mass (by melting, evaporation, etc.) and approximate the atmospheric density  $\rho$  by

$$\rho = \rho(y) = \rho_0 e^{-\beta y},$$

where  $y$  is the altitude measured in cm,  $\beta = 750,000^{-1} \text{ cm}^{-1}$ , and  $\rho_0$  is the sea level density.

- (a) Find the ODE for  $V^2$  as a function of  $y$  and try to solve it.

(b) Determine the speed *history* of an object, i.e., its speed as a function of time, given that its speed is  $V_0$  at  $y = 3 \times 10^7$  cm. Typical values of  $V_0$  might be in the range  $3 \times 10^5$  cm/sec  $< V_0 < 3 \times 10^6$  cm/sec.

4. Consider the Laplace equation,  $u_{xx} + u_{yy} = 0$ , in the interior of the unit circle,  $x^2 + y^2 < 1$ , where  $u = u(x, y)$ . Suppose that  $u$  satisfies the boundary condition

$$\frac{\partial u}{\partial \rho} = 0$$

along the circle  $x^2 + y^2 = 1$ , where  $(\rho, \phi)$  are the usual polar coordinates in the plane. Is this boundary value problem (BVP) ill posed or well posed? Explain.

5. (Levine, Chap. 1, Prob. 7, p. 14.) This problem is an exercise in partial differentiation. Verify by a direct calculation that the function  $u(x, t)$  given by

$$u(x, t) = \frac{1}{2}c \operatorname{sech}^2 \left[ \frac{1}{2} \sqrt{c}(x - ct) \right],$$

where  $c$  is a constant parameter, satisfies the PDE

$$u_t + 6uu_x + u_{xxx} = 0.$$

**Remark (not needed to solve the problem):** This solution is the celebrated *solitary wave* that moves at speed  $c$  in direction  $x$  *without changing its shape*;  $t$  is time. The governing PDE describes shallow water waves (KdV equation). This type of wave was first observed by John Scott Russell on the Edinburg-Glasgow Canal in 1834.

6. (Levine, Chap. 1, Prob. 1, p. 11.) By use of the equations  $x = \rho \cos \phi$  and  $y = \rho \sin \phi$  that relate the Cartesian coordinates  $(x, y)$  with the polar coordinates  $(\rho, \phi)$  ( $\rho \geq 0$  and  $0 \leq \phi < 2\pi$ ), express in polar coordinates and then solve the PDE

$$yu_x - xu_y = 0.$$

**Note:** The solution of PDE is often facilitated by coordinate transformations.

7. Find the general solution in Cartesian coordinates to the 1st-order PDE

$$yu_x - xu_y = x^2 + y^2.$$

Verify your solution by direct differentiation.

8. (Levine, Chap. 8, Prob. 5, p. 90.) (a) Find the general solution to the PDE

$$yu_x + xu_y = 2xy.$$

(b) Find the particular solution that satisfies  $u = y$  on the circle  $x^2 + y^2 = 1$ .

9. Determine the solution  $u = u(x, y)$  to the PDE

$$uu_x - uu_y = y - x$$

that satisfies  $u(1, y) = g(y)$ : known.

10. (Garabedian, Sec. 2.1, Prob. 2, pp. 22, 23.) Show that the solution of the nonlinear PDE  $u_x + u_y = u^2$  that satisfies  $u = x$  on the line  $y = -x$  becomes infinite along the hyperbola  $x^2 - y^2 = 4$ .